

# NEGATIVE POINT MASS SINGULARITIES IN GENERAL RELATIVITY

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Dissertation submitted in partial fulfillment of the  
requirements for the degree of Doctor of Philosophy  
in the Department of Mathematics  
in the Graduate School of  
Duke University

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ABSTRACT

(Mathematics)

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# Abstract

First we review the definition of a negative point mass singularity. Then we examine the gravitational lensing effects of these singularities in isolation and with shear and convergence from continuous matter. We review the Inverse Mean Curvature Flow and use this flow to prove some new results about the mass of a singularity, the ADM mass of the manifold, and the capacity of the singularity. We describe some particular examples of these singularities that exhibit additional symmetries.

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# Chapter 1

## Introduction

General Relativity is currently the accepted model of the physics of the universe on large scales. The theory principally consists of three parts.

- The universe is modeled by a four dimensional Lorentzian manifold.
- The geometry of this manifold is given by the Einstein equation:

$$Rc_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (1.1)$$

The left hand side is called the Einstein Tensor and the right hand side is the stress energy tensor for the system. The stress energy tensor is a property of the matter in the system.

- Particles, in the absence of other forces, move on timelike geodesics.

The theory has had remarkable success. Its major accomplishments include explaining the orbital perihelion precession of Mercury, differing values for the bending of light near a massive body, and a working cosmological model running up to moments after the Big Bang.

In particular, Gravitational Lensing has provided the most useful method for observing and studying that portion of the matter in the universe that does not emit or absorb light. This is currently believed to comprise approximately 85% of the total matter in the universe.

Despite its many successes, there are a number of difficulties that arise when working in the theory. The Einstein Equation, even in a vacuum, is a nonlinear second order hyperbolic differential equation in the metric. This makes exact solutions difficult to find. While the behavior of small test particles is easily given by the geodesic condition, the behavior of continuous masses, with features like tension, pressure, et cetera, are not given by the theory, but require an external derivation. Furthermore the relevant manifold is Lorentzian, not Riemannian, removing many powerful tools.

To avoid many of these difficulties, one may study Riemannian General Relativity. This is the study of Riemannian 3-manifolds that could arise as spacelike hypersurfaces in a spacetime in General Relativity. The Einstein Equation is translated into equations about the metric on this spacelike hypersurface and its second fundamental form. The properties of the matter in the theory are replaced by conditions such as the Dominant and Weak Energy conditions, which are also conditions on the metric of the hypersurface and its second fundamental form. Many of the important questions in General Relativity have analogues on Riemannian General Relativity. For example, the Penrose Conjecture can be restricted to the Riemannian Penrose Inequality.

This thesis consists of a study of the properties of Negative Point Mass Singularities. The motivating example of which is the spacial Schwarzschild metric with a negative mass parameter:

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} \quad m < 0. \quad (1.2)$$

In addition to being historically and physically important, the Schwarzschild solution is of particular mathematical interest since it is the case of equality of the Riemannian Penrose conjecture, and, in the case when  $m = 0$ , it is the case of equality of the Riemannian Positive Mass Theorem. Thus this metric, and its generalizations, show promise as objects of study.

This thesis consists of three main topics. After laying out the necessary definitions in Chapter 2, we examine the gravitational lensing effects of these singularities in Chapter 3. Next we summarize the work of Huisken and Ilmanen on Inverse Mean Curvature Flow in Chapter 4. We then use this to prove a number of results in Chapter 5. In Chapter 6 we show what additional information can be gained if the singularities possess additional symmetries.

# Chapter 2

## Definitions

### 2.1 Asymptotically Flat Manifolds

Physically, we want to make sure our manifolds represent isolated systems. A precise formulation of this is asymptotic flatness.

**Definition 2.1.1.** ([7]) A Riemannian 3-manifold  $(M, g)$  is called *asymptotically flat* if it is the union of a compact set  $K$ , and sets  $E_i$  diffeomorphic to the complement of a compact set  $K_i$  in  $\mathbb{R}^3$ , where the metric of each  $E_i$  satisfies

$$|g_{ij} - \delta_{ij}| \leq \frac{C}{|x|}, \quad |g_{ij,k}| \leq \frac{C}{|x|^2} \quad (2.1)$$

as  $|x| \rightarrow \infty$ . Derivatives are taken in the flat metric  $\delta_{ij}$  on  $x \in \mathbb{R}^3$ . Furthermore the Ricci curvature must satisfy

$$\text{Rc} \geq -\frac{Cg}{|x|^2}. \quad (2.2)$$

The set  $E_i$  is called an *end* of  $M$ .

A manifold may have several ends, but most of our results will be relative to a single end.

In [1], Arnowitt, Deser and Misner define a geometric invariant that is now called the ADM mass.

**Definition 2.1.2.** The *ADM mass* of an end of an asymptotically flat manifold is

$$m_{\text{ADM}} = \lim_{r \rightarrow \infty} \frac{1}{16\pi} \int_{S_\delta^r} (g_{ij,i} - g_{ii,j}) n^j d\mu. \quad (2.3)$$

This quantity is finite exactly when the total scalar curvature of the chosen end is finite. This definition appears to be coordinate dependent, however in [1] the authors show that it is actually an invariant when

$$\int_{M \setminus K} |R| < \infty. \quad (2.4)$$

## 2.2 Quasilocal Mass Functionals

While the ADM mass provides a definition for the total mass of a manifold, or the mass seen at infinity, there is no computable definition for the mass of a region. The two of most relevance are the (Riemannian) Hawking mass and the Bartnik mass.

**Definition 2.2.1.** The *Hawking mass* of a surface  $\Sigma$  is given by

$$m_{\text{H}} = \sqrt{\frac{|\Sigma|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma} H^2 \right). \quad (2.5)$$

Consider the Hawking mass of a surface,  $\Sigma$ , in  $\mathbb{R}^3$ , and note that  $H = \kappa_1 + \kappa_2$ , where  $\kappa_i$  are the principal curvatures of  $\Sigma$ . Thus  $H^2 = \kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2$ . Hence, for a sphere in  $\mathbb{R}^3$ ,

$$\int_{\Sigma} H^2 = \int_{\Sigma} 2K + \kappa_1^2 + \kappa_2^2 \geq \int_{\Sigma} 2K + 2K = 8\pi\chi(\Sigma) = 16\pi.$$

Thus the Hawking mass of a sphere is always nonpositive in  $\mathbb{R}^3$ . Furthermore the Hawking mass can be decreased by making the surface  $\Sigma$  have high frequency oscillations. These two observations lead to the conclusion that the Hawking mass tends to underestimate the mass in a region.

The other quasilocal mass functional of interest is the Bartnik mass defined in [2].

**Definition 2.2.2.** Let the asymptotically flat manifold  $(M, g)$  have nonnegative scalar curvature. Let  $\Omega$  be a domain in  $M$  with connected boundary. Assume  $M$  has no horizons (minimal spheres) outside of  $\Omega$ . Call any asymptotically flat manifold  $(\widetilde{M}, \widetilde{g})$  acceptable if it has nonnegative scalar curvature, contains an isometric copy of  $\Omega$ , and has no horizons outside of  $\Omega$ . Then the *Bartnik mass*,  $m_B(\Omega)$  of  $\Omega$  is defined to be the infimum of the ADM masses of these acceptable manifolds.

The positive mass theorem guarantees that this mass will be positive if the interior of  $\Sigma$  fulfills the hypotheses of the theorem. The Bartnik mass is very difficult to compute. The only cases where it is known are when the surface can be embedded in the exterior region of the Schwarzschild spacial metric or in  $\mathbb{R}^3$ . The Schwarzschild metric is the case of equality of the Riemannian Penrose inequality and  $\mathbb{R}^3$  is the case of equality for the positive mass theorem.

## 2.3 Definition and Mass of Negative Point Mass Singularities

The basic example of a negative point mass singularity is the negative Schwarzschild solution. This is the manifold  $\mathbb{R}^3 \setminus B_{-m/2}$  with the metric

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} \quad (2.6)$$

where  $m < 0$ . This manifold fails the requirements of the positive mass theorem since it is not complete: geodesics reach the sphere at  $r = -m/2$  in finite distance. A straightforward calculation shows that the ADM mass of this manifold is given by  $m$ . Furthermore the far field deflection of geodesics is the same as for a Newtonian mass of  $m$ . These results are identical to the same results for a positive mass Schwarzschild solution.

Two important aspects of this example will be incorporated into the definition of a negative point mass singularity. One is that the point itself is not included. To justify the use of the word “point” we have to describe the behavior of surfaces near the singularity. The manifold in that region should have surfaces whose areas converge to zero. In addition the capacity of these surfaces should go to zero. The second aspect is the presence of a background metric, in this case the flat metric. This background metric will provide a location where we can compute information about the singularity.

This example motivates the following definition.

**Definition 2.3.1.** Let  $M^3$  be a smooth manifold with boundary, where the boundary is compact. Let  $\Pi$  be a compact connected component of the boundary of  $M$ . Let

the interior of  $M$  be a Riemannian manifold with smooth metric  $g$ . Suppose that, for any smooth family of surfaces which locally foliate a neighborhood of  $\Pi$ , the areas with respect to  $g$  go to zero as the surfaces converge to  $\Pi$ . Then  $\Pi$  is a *Negative Point Mass Singularity*.

A particularly useful class of these singularities are Regular Negative Point Mass Singularities.

**Definition 2.3.2.** Let  $M^3$  be a smooth manifold with boundary. Let the boundary of  $M$  consist of one compact component,  $\Pi$ . Let  $(M^3 \setminus \Pi, g)$  be a smooth Riemannian manifold. Suppose that, for any smooth family of surfaces which locally foliate a neighborhood of  $\Pi$ , the areas with respect to  $g$  go to zero as the surfaces converge to  $\Pi$ . If there is a smooth metric  $\bar{g}$  on  $M^3$  and a smooth function  $\bar{\varphi}$  on  $M$  with nonzero differential on  $\Pi$  so that  $g = \bar{\varphi}^4 \bar{g}$ , then we call  $\Pi$  a *Regular Negative Point Mass Singularity*. We call the data  $(M^3, \bar{g}, \bar{\varphi})$  a *resolution* of  $\Pi$ .

Notice that while  $\Pi$  is topologically a surface, and it is a surface in the Riemannian manifold  $(M^3, \bar{g})$ , the areas of surfaces near it in  $(M^3 \setminus \Pi, g)$  approach zero, so we will sometimes speak of  $\Pi$  as being a point  $p$ , when we are thinking in terms of the metric  $g$ . Furthermore, notice that the requirement that areas near  $\Pi$  go to zero under  $g$  tells us that  $\bar{\varphi} = 0$  on  $\Pi$ .

We can define the mass of a Regular Negative Point Mass Singularity as follows:

**Definition 2.3.3.** Let  $(M^3, \bar{g}, \bar{\varphi})$  be a resolution of a regular negative point mass singularity  $p = \Pi$ . Let  $\bar{\nu}$  be the unit normal to  $\Pi$  in  $\bar{g}$ . If the capacity of  $p$  is zero,

then the *regular mass* of  $p$  is defined to be

$$m_{\text{R}}(p) = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Pi} \overline{\nu}(\overline{\varphi})^{4/3} \overline{dA} \right)^{3/2}. \quad (2.7)$$

If the capacity of  $p$  is nonzero, then the mass of  $p$  is defined to be  $-\infty$ .

See Chapter 5 for a discussion of the capacity of points like  $p$ . We can also define the mass of a negative point mass singularity that may not be regular.

**Definition 2.3.4.** Let  $(M^3, g)$  be an asymptotically flat manifold, with a negative point mass singularity  $p$ . Let  $\Sigma_i$  be a smooth family of surfaces converging to  $p$ . Define  $h_i$  by

$$\Delta h_i = 0 \quad (2.8)$$

$$\lim_{x \rightarrow \infty} h_i = 1 \quad (2.9)$$

$$h_i = 0 \text{ on } \Sigma_i. \quad (2.10)$$

Then the manifold  $(M, h_i^4 g)$  has a negative point mass singularity at  $\Sigma_i = p_i$  which is resolved by  $(M, g, h_i)$ . Define the mass of  $p$  to be

$$\sup_{\{\Sigma_i\}} \overline{\lim}_{i \rightarrow \infty} -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma_i} \nu(h_i)^{4/3} dA \right)^{3/2} = \sup_{\{\Sigma_i\}} \overline{\lim}_{i \rightarrow \infty} m_{\text{R}}(p_i). \quad (2.11)$$

Here the outer sup is over all possible smooth families of surfaces  $\{\Sigma_i\}$  which converge to  $p$ .

A straightforward calculation shows that if the capacity of  $p$  is non-zero, then the mass of  $p$  is  $-\infty$ . This is the definition we will be working with. However, an alternative definition for the mass of a negative point mass singularity follows.

**Definition 2.3.5.** Let  $(M^3, g)$  be an asymptotically flat manifold, with a negative point mass singularity  $p$ . Choose a function  $h$  that satisfies the equations

$$\Delta h = 0 \tag{2.12}$$

$$h = \frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{2.13}$$

$$\lim_{x \rightarrow p} h = \infty. \tag{2.14}$$

Now define surfaces  $\Sigma_t = \{x | h(x) = t\}$  and functions  $\varphi_t(x) = 1 - h/t$ . Then the manifold  $(M, \varphi_t^4 g)$  has a negative point mass singularity at  $\Sigma_t = p_t$  which is resolved by  $(M, g, \varphi_t)$ . Define the mass of  $p$  to be

$$\sup_h \overline{\lim}_{t \rightarrow \infty} -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma_t} \nu(\varphi_t)^{4/3} dA \right)^{3/2} = \sup_h \overline{\lim}_{t \rightarrow \infty} m_R(p_t). \tag{2.15}$$

Here the outer sup is over all possible  $h$ 's.

## 2.4 Fundamental Results

Before we continue we must verify that these definitions are consistent. First it must be verified that the regular mass of a regular singularity is indeed intrinsic to the singularity, as shown in [6].

**Lemma 2.4.1.** *The regular mass of a negative point mass singularity is independent of the resolution.*

*Proof.* Let  $(M^3, \bar{g}, \bar{\varphi})$  and  $(M^3, \tilde{g}, \tilde{\varphi})$  be two resolutions of the same negative point

mass singularity,  $p$ . Then define  $\lambda$  by  $\bar{\varphi} = \lambda \tilde{\varphi}$ . Thus we note the following scalings:

$$\tilde{g} = \lambda^4 \bar{g} \quad (2.16)$$

$$\widetilde{dA} = \lambda^4 \overline{dA} \quad (2.17)$$

$$\tilde{\varphi} = \lambda^{-1} \bar{\varphi} \quad (2.18)$$

$$\tilde{\nu} = \lambda^{-2} \bar{\nu} \quad (2.19)$$

Now note that since  $\tilde{\varphi}, \bar{\varphi} = 0$  on  $\tilde{\Pi}, \bar{\Pi}$ ,

$$\tilde{\nu}(\tilde{\varphi}) = \lambda^{-2} \bar{\nu}(\lambda^{-1} \bar{\varphi}) = \lambda^{-3} \bar{\nu}(\bar{\varphi}) + \lambda^{-4} \bar{\nu}(\lambda) \bar{\varphi}. \quad (2.20)$$

The last term,  $\lambda^{-4} \bar{\nu}(\lambda) \bar{\varphi}$ , needs discussion. Both  $\bar{\varphi}$  and  $\tilde{\varphi}$  are smooth functions with zero set  $\Pi$  and they both have nonzero differential on  $\Pi$ . Thus  $\lambda$  is smooth. To see this choose a coordinate patch on the boundary where  $\Pi$  is given by  $x = 0$ . Then Taylor's formula tells us that

$$\lambda = \frac{\int_0^1 \frac{\partial \bar{\varphi}}{\partial x}(xs, y, z) ds}{\int_0^1 \frac{\partial \tilde{\varphi}}{\partial x}(xs, y, z) ds}, \quad (2.21)$$

which is a nonzero smooth function. Thus since  $\bar{\varphi}$  goes to zero on  $\Pi$ , this last term is zero on  $\Pi$ . Thus the mass of  $p$  using the  $(M^3, \tilde{g}, \tilde{\varphi})$  resolution is given by

$$m_{\text{R}}(p) = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\tilde{\Pi}} \tilde{\nu}(\tilde{\varphi})^{4/3} \widetilde{dA} \right)^{3/2} \quad (2.22)$$

$$= -\frac{1}{4} \left( \frac{1}{\pi} \int_{\bar{\Pi}} [\lambda^{-2} \bar{\nu}(\lambda^{-1} \bar{\varphi})]^{4/3} \lambda^4 \overline{dA} \right)^{3/2} \quad (2.23)$$

$$= -\frac{1}{4} \left( \frac{1}{\pi} \int_{\bar{\Pi}} [\lambda^{-3} \bar{\nu}(\bar{\varphi})]^{4/3} \lambda^4 \overline{dA} \right)^{3/2} \quad (2.24)$$

$$= -\frac{1}{4} \left( \frac{1}{\pi} \int_{\bar{\Pi}} \bar{\nu}(\bar{\varphi})^{4/3} \overline{dA} \right)^{3/2}. \quad (2.25)$$

□

Definition 2.3.4 seems to involve the entire manifold, as the definition of  $h_i$  takes place on the entire manifold. However that isn't the case. The mass is actually local to the point  $p$ .

**Lemma 2.4.2.** *Let  $(M^3, g)$  be a manifold with a negative point mass singularity  $p$ . Let  $\tilde{g}$  be a second metric on  $M$  that agrees with  $g$  in a neighborhood of  $p$ . Then the mass of  $p$  in  $(M^3, g)$  and  $(M^3, \tilde{g})$  are equal.*

*Proof.* The goal is to show that for any selection of  $\{\Sigma_i\}$ , the series  $m_R(p_i)$  and  $\tilde{m}_R(p_i)$  obtained in the calculation of the mass of  $p$ , with respect to  $(M, g)$  and  $(M, \tilde{g})$  converge. Let  $S$  be a smooth, compact, connected surface separating  $p$  from infinity and contained in the region where  $g$  and  $\tilde{g}$  agree. Fix  $i$  large enough so that  $\Sigma_i$  is inside of  $S$ , and suppress the index  $i$  on all our functions. Then define the functions  $h, \tilde{h}$  by

$$h = \tilde{h} = 0 \text{ on } \Sigma_i$$

$$\lim_{x \rightarrow \infty} h = \lim_{x \rightarrow \infty} \tilde{h} = 1$$

$$\Delta h = \tilde{\Delta} \tilde{h} = 0.$$

Here  $\Delta$  and  $\tilde{\Delta}$  denote the Laplacian with respect to  $g$  and  $\tilde{g}$  respectively.

Now inside  $S$ ,  $\Delta = \tilde{\Delta}$  since  $g = \tilde{g}$ . Thus there is only one notion of harmonic, and  $h$  and  $\tilde{h}$  differ only by their boundary values on  $S$ . Let  $\epsilon = 1 - \min_S \{h, \tilde{h}\}$ .

Consider the following two functions  $f^-$  and  $f^+$  defined between  $S$  and  $\Sigma_i$ :

$$f^- = f^+ = 0 \text{ on } \Sigma_i$$

$$\Delta f^- = \Delta f^+ = 0$$

$$f^- = 1 - \epsilon \text{ on } S$$

$$f^+ = 1 \text{ on } S.$$

Thus by the maximum principle, we have the following inside  $S$

$$f^+ \geq h, \tilde{h} \geq f^-. \quad (2.26)$$

Furthermore, since all four functions are zero on  $\Sigma_i$ ,

$$\nu(f^+) \geq \nu(h), \nu(\tilde{h}) \geq \nu(f^-). \quad (2.27)$$

Here  $\nu$  is the normal derivative on  $\Sigma_i$ . Now define  $\mathcal{E}(\varphi)$  by the formula

$$\mathcal{E}(\varphi) = \int_{\Sigma_i} \nu(\varphi)^{4/3} dA. \quad (2.28)$$

Then the ordering of the derivatives gives the ordering

$$\mathcal{E}(f^+) \geq \mathcal{E}(h), \mathcal{E}(\tilde{h}) \geq \mathcal{E}(f^-). \quad (2.29)$$

However, since  $f^- = (1 - \epsilon)f^+$ ,

$$\nu(f^-) = (1 - \epsilon)\nu(f^+), \quad (2.30)$$

hence,

$$\mathcal{E}(f^-) = (1 - \epsilon)^{4/3} \mathcal{E}(f^+). \quad (2.31)$$

Now, without loss of generality assume that the limit of the capacities of  $\{\Sigma_i\}$  is zero, as the mass would be  $-\infty$  otherwise. Thus as  $i \rightarrow \infty$ ,  $\Sigma_i$  has capacity going to zero. Hence  $\epsilon_i$  goes to zero, and so  $\mathcal{E}(f_i^-)/\mathcal{E}(f_i^+)$  goes to 1. Thus equation (2.29) forces  $\mathcal{E}(h_i)$  and  $\mathcal{E}(\tilde{h}_i)$  to equality. This forces the masses of  $p_i$  in the two metrics to equality as well.  $\square$

**Corollary 2.4.3.** *In Definition 2.3.4 we may replace the condition that  $\varphi_i$  be one at infinity with the condition that  $\varphi_i$  be one on a fixed surface outside  $\Sigma_i$  for  $i$  sufficiently large.*

This mass also agrees with the regular mass when the singularity is regular.

**Lemma 2.4.4** ([3]). *Let  $(M^3, g)$  be an asymptotically flat manifold with negative point mass singularity  $p$ . Let  $p$  have a resolution  $(M^3, \tilde{g}, \tilde{\varphi})$ . Then the regular mass of  $p$  equals the general mass of  $p$ .*

# Chapter 3

## Gravitational Lensing by Negative Point Mass Singularities

### 3.1 Gravitational Lensing Background

One of the first testable predictions of general relativity was the difference in the deflection of light by gravity. This effect was first confirmed during the 1919 solar eclipse. Since then gravitational lensing has become an powerful tool for astronomy in general and cosmology in particular. Gravitational lensing has made it possible to detect the presence of dark matter by observing its effects on background images.

In this chapter we will develop the properties of lensing by negative point mass singularities in the setting of accepted cosmology. We will make a number of assumptions based on that cosmology that will allow us to obtain simple formulas for gravitational lensing by negative point mass singularities. Then we will characterize their lensing effects and compare them to positive mass point sources. We will not cover the entire field of gravitational lensing, but only develop enough for our purposes.

We will restrict ourselves to negative point mass singularities that agree with a negative Schwarzschild solution to first order. We will find that the lensing effects of these singularities in the presence of continuous matter and shear can be duplicated by configurations with positive mass lenses.

We will follow the presentation given in [9]. We will differ from this presentation by using geometrized units where  $c = G = 1$ . We will also consider lens potentials outside of the scope of [9].

### 3.2 Cosmology for Gravitational Lensing

To simplify the calculations involved in lensing, we will make use of a number of assumptions about the configuration of our system and its behavior. These assumptions are based on the scales and phenomenology of astronomy. Our first assumption is one of cosmology.

**Assumption 3.2.1.** The universe is described by a Friedmann–Robertson–Walker cosmology.

The Friedmann–Robertson–Walker model is an isotropic homogeneous cosmology filled with perfect dust. We will not develop this cosmology from these properties but merely take it as a given. In this cosmology, the universe is modeled as a warped product with leaves  $\mathbb{R}$  and fibers given by either  $\mathbb{R}^3$ ,  $H^3$  or  $S^3$ . Thus we have the metric

$$ds^2 = -d\tau^2 + a^2(\tau)dS_K^2. \quad (3.1)$$

Where  $S_K$  is  $H^3$ ,  $\mathbb{R}^3$ , or  $S^3$  when  $K = -1, 0, 1$ , respectively. The coordinate  $\tau$  is time as measured by the isotropic observers. This is called “cosmological” time. The

function  $a(\tau)$  gives the scale of the universe, and has the following relationship to  $\tau$  depending on  $K$

$$K = 1 \quad a = \frac{A}{2} (1 - \cos u) \quad \tau = \frac{A}{2} (1 - \sin u) \quad (3.2)$$

$$K = 0 \quad a = \left( \frac{9A}{4} \right)^{1/3} \tau^{2/3} \quad (3.3)$$

$$K = -1 \quad a = \frac{A}{2} (\cosh u - 1) \quad \tau = \frac{A}{2} (\sinh u - u). \quad (3.4)$$

The metric on the fibers is given by

$$dS_K^2 = \frac{dR^2}{1 - KR^2} + R^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (3.5)$$

If we write

$$R = \sin_K(\chi) = \begin{cases} \sin \chi & \text{if } \chi = 1 \\ \chi & \text{if } \chi = 0 \\ \sinh \chi & \text{if } \chi = -1, \end{cases} \quad (3.6)$$

then we can rewrite the fiber metric as

$$dS_K^2 = d\chi^2 + \sin_K^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (3.7)$$

The differences between all of these metrics are small on scales small compared to the size of the universe, where most of our calculations will take place. We also introduce a second time coordinate,  $t$ , by the following equation

$$t = \int \frac{d\tau}{a(\tau)}. \quad (3.8)$$

We can use this to rewrite the metric on the universe as

$$ds^2(t) = a^2(t) (-dt^2 + dS_K^2). \quad (3.9)$$

For this reason  $t$  is called “conformal time.”

In our discussion of the geometry of a lens system, we will need to discuss the distances between the observer, lens and source. There are a number of options available, depending on which equations one wants to simplify. We will use “angular diameter distance.” This distance is defined by the ratio the physical size of an object to the angular size of the object seen by the observer. We denote the distance from the observer to the lens by  $d_L$ , the distance from lens to source by  $d_{L,S}$ , and the distance from observer to source by  $d_S$ . For lensing on small scales (nearby galaxies) the universe is almost flat, and since the bending angle is generally small, we can use  $d_S \simeq d_{L,S} + d_L$ .

We will also need the redshift. As the universe expands, the wavelength of light from distant sources is lengthened. This is equivalent to clocks appearing to run more slowly. This is properly associated to an event, but since it changes slowly compared to the size and duration of a lensing event, we will be assuming it is locally constant. We can define the redshift of a time,  $\tau$ , (or an event) by

$$z = \frac{a(\tau_O)}{a(\tau)} - 1. \quad (3.10)$$

### 3.3 Geometry of Lens System

We make a number of assumptions about the geometry of a gravitational lensing system. First we assume that the lens isn’t changing on the time scale it takes for light to cross the lens. This is valid since most objects evolve at speeds much less than the speed of light. We can also assume that the lens is stationary. Any relative motion will be attributed to the source.

**Assumption 3.3.1.** The geometry of the spacetime is assumed to be unchanging

on the time scale of the lensing event.

Furthermore, since almost all sources are “weak” we also assume that the entire system lies in the weak regime, where can approximate the metric by a time-independent Newtonian potential,  $\varphi$ , given by

$$\varphi(x) = -a_L^2 \int_{\mathbb{R}^3} \frac{\rho(\tilde{x})}{\|x - \tilde{x}\|} d\tilde{x}. \quad (3.11)$$

Here  $a_L$  is the value of  $a$  when the light ray is interacting with the lens.

**Assumption 3.3.2.** The spacetime metric of gravitational lens system is given by

$$ds^2 = -(1 + 2\varphi) d\tau^2 + a^2(\tau)(1 - 2\varphi) dS_K^2 \quad (3.12)$$

$$= a^2(t) \left[ -(1 + 2\varphi) dt^2 + (1 - 2\varphi) dS_K^2 \right]. \quad (3.13)$$

Furthermore,  $\varphi$ , the Newtonian potential, is much smaller than unity. We will also assume that while the light ray is interacting with the lens,  $a(t)$  is constant. Furthermore, since on the scale of this interaction, the universe is almost flat, we will assume  $K = 0$  during the interaction. Thus near the lens we have the following.

**Assumption 3.3.3.** During the interaction of the light ray with the lens, the metric of the lens can be assumed to be

$$ds_L^2 = a_L^2 \left[ -(1 + 2\varphi) dt^2 + (1 - 2\varphi) dS^2 \right]. \quad (3.14)$$

Furthermore,  $dt$  can be approximated by  $\frac{1}{a_L} d\tau$ .

We choose dimensionless Euclidean coordinates,  $(x_1, x_2, x_3)$ , centered at the lens so that  $dS^2 = \delta_{ij}$  and so that the line of sight to the lens is along the  $x_3$  axis. We will also use proper coordinates  $(r_1, r_2, \zeta) = a_L \cdot (x_1, x_2, x_3)$ , and will use the coordinates  $r = (r_1, r_2)$  on the lens plane. We will use proper coordinates  $s = (s_1, s_2)$  in the source plane.

### 3.3.1 Mass Densities, Bending Angle and Index of Refraction

Since we are in the static weak field limit, the Einstein equation reduces to the time independent Poisson equation for  $\varphi$ . Thus we get the three dimensional Poisson equation

$$\Delta^{3D}\varphi(x) = 4\pi a_L^2 \rho(x). \quad (3.15)$$

Where  $\rho$  is the density above the background of the lens. The solution to this equation is

$$\varphi(x) = -a_L^2 \int_{\mathbb{R}}^3 \frac{\rho dx'}{\|x - x'\|}. \quad (3.16)$$

Since we are assuming that the lens is planar, it is useful to project the three dimensional potential and density into the lens plane. Integrating  $\rho$  along the line of sight gives us the surface mass density of the lens  $\sigma(r)$ . This integral is really only from  $-d_{L,S}$  to  $d_L$ , but since we are assuming that  $\rho$  is zero except near the lens we can extend this integral to  $(-\infty, \infty)$ . Thus we get that

$$\sigma(r) = \int_{\mathbb{R}} \rho(r_1, r_1, \zeta) d\zeta. \quad (3.17)$$

Integrating the three dimensional Poisson equation along the  $\zeta$  axis gives us

$$4\pi\sigma(r) = \int_{\mathbb{R}} \left( \frac{\partial^2 \varphi}{\partial r_1^2} + \frac{\partial^2 \varphi}{\partial r_2^2} + \frac{\partial^2 \varphi}{\partial \zeta^2} \right) d\zeta = \Delta^{2D} \int_{\mathbb{R}} \varphi(r_1, r_2, \zeta) d\zeta. \quad (3.18)$$

The  $\frac{\partial^2 \varphi}{\partial \zeta^2}$  term integrates to zero since  $\varphi$  is zero when  $\zeta = \pm\infty$ . If we define the surface potential of the lens by

$$\Psi(r) = 2 \int_{\mathbb{R}} \varphi(r_1, r_2, \zeta) d\zeta, \quad (3.19)$$

then  $\Psi$  satisfies the two dimensional Poisson equation

$$\Delta^{2D}\Psi(r) = 8\pi\sigma(r). \quad (3.20)$$

This is solved by

$$\Psi(r) = 4 \int_{\mathbb{R}^2} \sigma(r') \ln \left\| \frac{r - r'}{d_0} \right\| dr', \quad (3.21)$$

for any constant  $d_0$ . We will generally choose  $d_0 = d_L$ .

It will become useful to think of the potential of the lens giving a refractive index to the spacetime. The index of refraction of a medium is the reciprocal of the velocity of light in that medium. We want to calculate the velocity of light in our metric,  $ds_L^2$ , relative to the flat metric given by  $a_L^2 \delta_{ij}$ . The velocity of light in the metric

$$ds^2 = -A(x)dt^2 + B(x)dS^2 \quad (3.22)$$

is given by the ratio

$$n = \sqrt{\frac{B}{A}}. \quad (3.23)$$

In the metric  $ds_L^2$  we get

$$n = \sqrt{\frac{1-2\varphi}{1+2\varphi}} \simeq 1 - 2\varphi \quad (3.24)$$

to first order in  $\varphi$ . Since  $\varphi$  is much smaller than unity, we can ignore higher order terms.

Now we can look at the bending angle of the lens. We approximate the light ray from the source to the observer by a broken null geodesic with the break at the lens plane. We compress all the bending in the ray due to the lens into this corner of the light ray. Call the tangent to the incoming light ray  $T_i(r)$  and tangent to the final ray  $T_f(r)$ . Then we define the bending angle by

$$\hat{\alpha}(r) = T_f(r) - T_i(r). \quad (3.25)$$

Here we have parametrized the vectors by the impact parameter  $r$ . Continuing with the standard geometric optics approximation, we parametrize the spatial path  $R(s) = (R_1(s), R_2(s), R_3(s))$  of the light ray by arclength,  $s$ , in the background metric ( $ds^2 = a_L^2 \delta_{ij}$ ). Then light rays are characterized by the equation

$$\frac{d}{ds} \left( n \frac{dR}{ds} \right) = \nabla n. \quad (3.26)$$

Here  $\nabla$  is the flat gradient. Now define the quantities  $T = \frac{dR}{ds}$  and  $K = \frac{dT}{ds}$  as the tangent and curvature vectors of the curve  $R(s)$ . Plugging these into equation (3.26) gives the equation

$$(Tn)T + nK = \nabla n. \quad (3.27)$$

Since  $K$  is perpendicular to  $T$  the transverse gradient is given by

$$\nabla_{\perp} n = nK. \quad (3.28)$$

Solving this for  $K$  gives us

$$K = \frac{\nabla_{\perp} n}{n} \simeq (-2\nabla_{\perp} \varphi) (1 - 2\varphi)^{-1} \simeq -2\nabla_{\perp} \varphi \quad (3.29)$$

to first order in  $\varphi$ . Since this angle is small, the light rays are almost perpendicular to the lens plane, so we can replace  $\nabla_{\perp}$  with the gradient in the  $(r_1, r_2)$  plane,  $\nabla_r$ , and we can integrate over  $\zeta$  to find the total  $K$  for the entire light ray. This total  $K$  tells us how far the tangent vector has turned, hence the bending angle is

$$\hat{\alpha}(r) = 2 \int \nabla_r \varphi(r_1, r_2, \zeta) d\zeta. \quad (3.30)$$

Pushing the integral inside the gradient gives us

$$\hat{\alpha}(r) = \nabla \Psi(r). \quad (3.31)$$

### 3.3.2 Fermat's Principle and Time Delays

One could use equation (3.31) to try and work out the effect of a lens, but instead we will follow [9] and use the following principle.

**Proposition 3.3.4** (Fermat's Principle). *A light ray from a source (an event) to an observer (a timelike curve) follows a path that is a stationary value of the arrival time functional,  $\mathcal{T}$ , on paths.*

Here we are only considering paths  $v_r$ , that are broken geodesics from the source  $S$ , to the observer  $O$ , parametrized by the impact parameter  $r = (r_1, r_2)$  where the ray crosses the lens plane. For a given source location  $s$  in the source plane, we look at the time delay function  $\mathcal{T}_s(r)$  that gives the time delay for a light ray that goes from  $s$  to  $r$ , bends at  $r$ , and then continues on to  $O$ . Technically the time delay function and the arrival time function differ by some reference value. That reference value is the time the ray would have taken in the absence of the lens. We denote this unlensed references path by  $u_0$ . Fermat's principle tells us that the images of a source at  $s$  are given by solutions to the equation

$$\nabla_r \mathcal{T}_s(r) = 0. \quad (3.32)$$

To calculate this we will separate the time delay into two components. One is the effect of the longer path the light ray takes. This is called the *geometric time delay*,  $\mathcal{T}_g$ . We will drop the  $s$  for the moment. The other effect is due to time passing more slowly in a gravitational potential as seen by a distant observer. This is called the *potential time delay*,  $\mathcal{T}_p$ . The travel time for the unlensed ray is given by

$$\int_{u_0} a_L dl. \quad (3.33)$$

Where  $dl$  is given by the metric  $dS_K^2$ . The travel time for the lensed ray is given by

$$\int_{v_r} a_L n_L dl. \quad (3.34)$$

Hence the time delay is given by

$$\mathcal{T}^L(r) = \int_{v_r} a_L n_L dl - \int_{u_0} a_L dl. \quad (3.35)$$

Here the  $L$  attached to  $\mathcal{T}$  denotes the fact that these time delays are being measured at the lens plane. We will have to account for the redshift,  $z_L$ , of the lens. Thus we define the geometric and potential time delays by

$$\mathcal{T}_p^L = \int_{v_r} a_L (n_L - 1) dl \quad \mathcal{T}_g^L = a_L \left( \int_{v_r} dl - \int_{u_0} dl \right). \quad (3.36)$$

The potential time delay is easiest to calculate. Since the bending angle is small, we can approximate  $v_r$  by  $u_0$ . Using equations (3.24) and (3.19) we can calculate the potential time delay as

$$\mathcal{T}_p^L(r) = -\Psi(r). \quad (3.37)$$

The geometric time delay is more complicated. We will first calculate the geometric time delay assuming that  $K = 0$ , since we are calculating many quantities to first order, the geometric time delay will be the same for  $K = \pm 1$ . For a detailed treatment of  $K = \pm 1$ , see [9].

First we define the dimensionless lengths  $l_S$ ,  $l_L$ , and  $l_{L,S}$  as the lengths of the spatial projections of  $u_0$ , and the parts of  $v_r$  between the lens and observer and observer and source respectively. These lengths are measured in the metric  $dS_K^2$ . Thus the geometric time delay measured at the lens is given by

$$\mathcal{T}_g^L = a_L (l_L + l_{L,S} - l_S). \quad (3.38)$$

The law of cosines tells us

$$l_S^2 = l_L^2 + l_{L,S}^2 - 2l_L l_{L,S} \cos(\pi - \hat{\alpha}). \quad (3.39)$$

We can approximate  $\cos(\pi - \hat{\alpha})$  by  $-1 + \hat{\alpha}^2/2$  since  $\hat{\alpha}$  is small. This gives us

$$l_S^2 \simeq l_L^2 + l_{L,S}^2 + 2l_L l_{L,S}(1 - \hat{\alpha}^2/2) \quad (3.40)$$

$$= (l_L + l_{L,S})^2 - l_L l_{L,S} \hat{\alpha}^2. \quad (3.41)$$

Isolating the term with  $\hat{\alpha}$  and factoring gives us

$$(l_L + l_{L,S} - l_S)(l_L + l_{L,S} + l_S) \simeq l_L l_{L,S} \hat{\alpha}^2 \quad (3.42)$$

$$\mathcal{T}_g^L \simeq \frac{l_L l_{L,S}}{l_L + l_{L,S} + l_S} \hat{\alpha}^2 \simeq \frac{l_L l_{L,S}}{2l_S} \hat{\alpha}^2. \quad (3.43)$$

We can replace  $l_L$  and  $l_{L,S}$  by  $a_L d_L$ , and  $a_L d_{L,S}$  respectively. We would like to remove the reference to  $\hat{\alpha}$ . To do that we first construct the point  $s'$  in the source plane. It is the location that would produce an image at  $r$  in the absence of the lens. Then using similar triangles we compute

$$\hat{\alpha} d_{L,S} = \|s' - s\| = \left\| \frac{r}{d_L} - \frac{s}{d_S} \right\| d_S. \quad (3.44)$$

Plugging this into equation (3.43) gives us

$$\mathcal{T}_g^L(r) \simeq \frac{1}{a_L} \frac{d_L d_S}{2d_{L,S}} \left\| \frac{r}{d_L} - \frac{s}{d_S} \right\|^2. \quad (3.45)$$

Adding the two parts of the time delay together, and correcting for the redshift by multiplying by  $1 + z_L$  gives us

$$\mathcal{T}_s(r) = (1 + z_L) \frac{d_L d_S}{d_{L,S}} \left( \frac{1}{2} \left\| \frac{r}{d_L} - \frac{s}{d_S} \right\|^2 - \frac{d_{L,S}}{d_L d_S} \Psi(r) \right). \quad (3.46)$$

Taking the gradient by  $r$  and solving for  $s$  gives the *lens equation*

$$s = \frac{d_S}{d_L} r - d_{L,S} \hat{\alpha}(r). \quad (3.47)$$

Viewing  $s$  as a function of  $r$  results in the *lensing map*. This map goes from the image plane to the source plane. It answers the question: “Where would a source have to be create an image at this location?” To further simplify this equation, we will nondimensionalize by introducing the following variables

$$x = r/d_L \quad y = s/d_S \quad (3.48)$$

$$\psi(x) = \left( \frac{d_{L,S}}{d_L d_S} \right) \Psi(r) \quad \alpha(x) = \frac{d_{L,S}}{d_S} \hat{\alpha}(r) \quad (3.49)$$

$$\kappa(x) = \frac{\sigma(r)}{\sigma_c} \text{ where } \sigma_c = \frac{d_S}{2\pi d_L d_{L,S}}. \quad (3.50)$$

With these variables, the lensing map becomes

$$y = \eta(x) = x - \alpha(x) \quad (3.51)$$

with  $\alpha = \nabla\psi$ .

### 3.3.3 Magnification

In addition to changing the location of the image of a source, gravitational lensing can also change the apparent size of an object. Since all sources are not truly point sources, we can really consider how a region,  $R$ , around a point  $s$  in the source plane gets deformed. In particular, the signed area of the image of  $R$  will be determined by the integral of the determinant of  $d\eta^{-1}$ . Due to the Brightness Theorem the apparent surface brightness of an object is invariant for all observers. For instance if one were twice as far from the sun, the total light received would be reduced by four, as would the area of the sun. So the observed surface brightness would remain constant. Thus the total light received at the observer from an extended source is scaled the same as the areas. See [11] for more information.

However, while many sources aren't point sources they are point-like. Thus the only effect of the magnification is to increase (or decrease) the brightness of the source. Thus the magnification of a point source at the location  $y$  is given by

$$\mu_y(x) = \frac{1}{|\det d\eta(x)|}. \quad (3.52)$$

Locations,  $x$ , in the lens plane where this is infinite are called *critical points*. The corresponding locations  $\eta(x) = y$  in source plane are called *caustics*. Sources on different sides of caustics typically have two fewer or less images than each other. It is often useful to consider a source that is moving in the source plane. As this source crosses the caustic, two of its images will increase in brightness and merge, then disappear, or the reverse depending on the direction in which the source crosses the caustic. The changing magnification of a source as it moves in the source plane is a useful observable called the *light curve*. For these curves, we add up the magnification of all the images, since sometimes the images are too close to be resolved.

### 3.4 Lensing Map for Isolated Negative Point Mass Singularities

The general framework we have established for gravitational lensing requires a potential function to plug into the metric in assumption 3.3.3 and following formulas. We will be studying singularities that agree to first order with negative Schwarzschild solutions. This is summarized in the following assumption:

**Assumption 3.4.1.** The two dimensional surface mass density for a negative point mass singularity is given by

$$\rho(r) = M\delta(r). \quad (3.53)$$

To first order, this assumption gives the correct behavior of a negative mass Schwarzschild solution in the weak field regime. We will restrict ourselves to cases that agree with this case to first order.<sup>1</sup>

We nondimensionalize our potential by defining

$$m = \frac{M}{\pi d_L^2 \sigma_c}. \quad (3.54)$$

Which gives us the dimensionless surface mass density

$$\kappa(x) = \pi m \delta(x). \quad (3.55)$$

This gives us the dimensionless surface potential

$$\psi(x) = m \ln(\|x\|), \quad (3.56)$$

and the dimensionless bending angle

$$\alpha(x) = m \frac{x}{\|x\|^2}. \quad (3.57)$$

Thus the dimensionless lensing map is given by

$$y = \eta(x) = x \left( 1 - \frac{m}{\|x\|^2} \right). \quad (3.58)$$

In this case, we can solve this equation exactly to find the images of a source at  $y$ .

$$x_{\pm} = \frac{1}{2} \left( \|y\| \pm \sqrt{\|y\|^2 + 4m} \right) \hat{y}. \quad (3.59)$$

Here  $\hat{y}$  is the unit vector in the direction of  $y$  from the origin. As long as  $\|y\| > 2\sqrt{-m}$ , we get two images for each source. Both images are on the same side of the

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<sup>1</sup>For a more detailed assumption of the positive mass analogue of this assumption see the discussion of point masses on p. 101 in [9].

lens. If we had  $m > 0$ , then our images would be on opposite sides of the lens. The derivative of the lensing map is given by

$$\begin{pmatrix} 1 - \frac{m}{\|x\|^2} + \frac{2x_1^2 m}{\|x\|^4} & \frac{2x_1 x_2 m}{\|x\|^4} \\ \frac{2x_1 x_2 m}{\|x\|^4} & 1 - \frac{m}{\|x\|^2} + \frac{2x_2^2 m}{\|x\|^4} \end{pmatrix}. \quad (3.60)$$

Hence the magnification is given by

$$\mu(x) = \frac{1}{1 - \frac{m^2}{\|x\|^4}} = \frac{\|x\|^4}{\|x\|^4 - m^2}. \quad (3.61)$$

For  $x_-$  this number is negative, so to get the total magnification we take the difference between signed magnifications<sup>2</sup>

$$\mu_{\text{tot}}(y) = \mu_y(x_+) - \mu_y(x_-) = \frac{\|y\|^2 + 2m^2}{\|y\| \sqrt{\|y\|^2 + 4m}}. \quad (3.62)$$

### 3.5 Light Curves for Isolated Negative Point Mass Singularities

As we calculated, the lensing map is given by

$$y = \eta(x) = x \left( 1 - \frac{m}{\|x\|^2} \right). \quad (3.63)$$

This has inverse

$$x_{\pm} = \frac{1}{2} \left( \|y\| \pm \sqrt{\|y\|^2 + 4m} \right) \hat{y}. \quad (3.64)$$

The inverse map tells us where an image will appear for a source located at  $y$  in the source plane. Since the radical is imaginary for  $\|y\| < 2\sqrt{-m}$ , these sources aren't visible at all. For sources outside this disk, we get two images:  $x_{\pm}$ . The reversed

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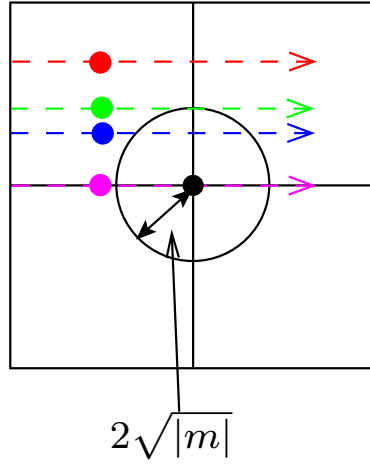
<sup>2</sup>See Appendix A.1 for the calculation.

image  $x_-$  is closer to the center of the lens. For large  $\|y\|$ , the  $x_-$  image is closer and closer to the center of the lens, while the  $x_+$  image is closer and closer to its unlensed location. As  $y$  gets closer and closer to the caustic  $\|y\| = 2\sqrt{-m}$ , the two images come together at  $x = \sqrt{-m}$ . Looking at the individual magnifications, we see that for large  $\|y\|$ , the positive image has magnification about 1, while the negative image has magnification about  $-m^2/\|y\|^4$ . As  $y$  gets closer and closer to the caustic, the magnification of each image increases, and is formally infinite when  $y = 2\sqrt{-m}$  and  $x_{\pm} = \sqrt{-m}$ . This curve is the critical curve. When  $y$  is inside the caustic, it creates no image.

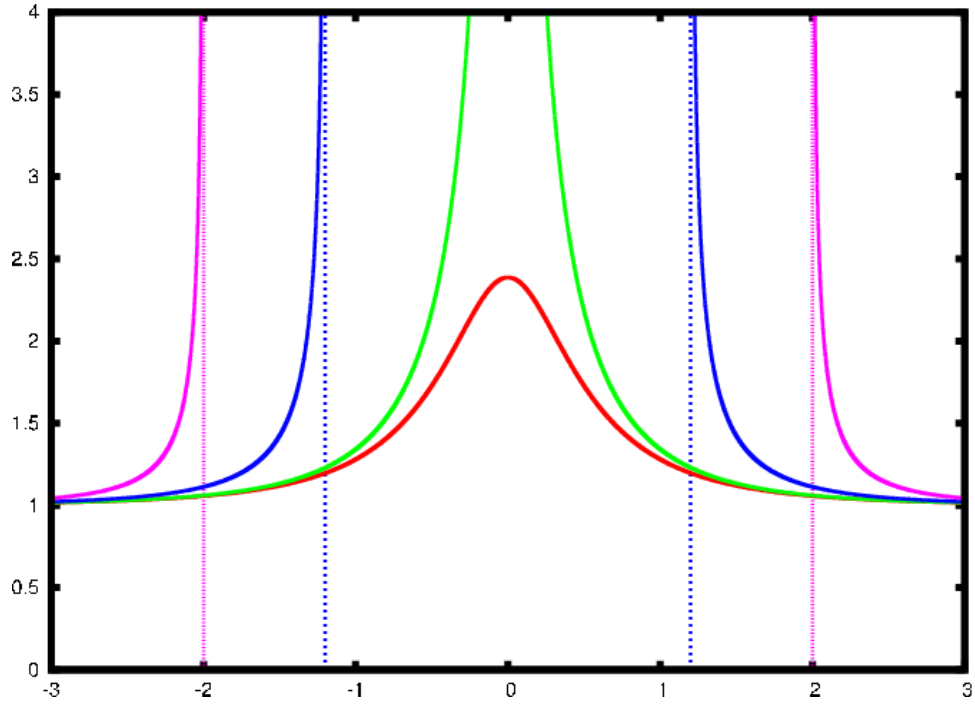
In cases where the source, lens and observer are moving relative to each other, the total magnification changes. We look at the total magnification, since the individual images are unresolvable, and the only effect of the lensing is the magnification. By the symmetry of the source, these paths are characterized by impact parameter, the distance of closest approach to  $y = 0$ . If this distance is  $d$ , and we assume that the source is moving at constant unit speed, the total magnification as a function of time is

$$\mu_{m,d}(t) = \frac{d^2 + t^2 + 2m}{\sqrt{d^2 + t^2}\sqrt{d^2 + t^2 + 4m}}. \quad (3.65)$$

Figure 3.1 shows several possible paths for a moving source. The sources in these curves have impact parameters varying from zero to twice  $2\sqrt{-m}$ . Figure 3.2 shows the corresponding light curves. The light curves for the sources passing inside of  $2\sqrt{-m}$  are distinctive, but those for the source passing outside or just along the caustic are not. The light curve formula, equation (3.65), is the same as that for a



**Figure 3.1:** Source Paths for Negative Point Mass Microlensing



**Figure 3.2:** Light Curves for Negative Point Mass Microlensing

positive point mass. Thus if we define

$$\tilde{m} = -m \quad \tilde{d} = \sqrt{d^2 + 4m}, \quad (3.66)$$

then the light curve for a source passing within  $\tilde{d}$  of the line of sight of a mass of  $\tilde{m}$  is the same as that for a source passing within  $d$  of the line of sight of a mass of  $m$

$$\mu_{\tilde{m}, \tilde{d}}(t) = \frac{\tilde{d}^2 + t^2 + 2\tilde{m}}{\sqrt{\tilde{d}^2 + t^2} \sqrt{\tilde{d}^2 + t^2 + 4\tilde{m}}} \quad (3.67)$$

$$= \frac{d^2 + 4m + t^2 - 2m}{\sqrt{d^2 + 4m + t^2} \sqrt{d^2 + 4m + t^2 - 4m}} \quad (3.68)$$

$$= \mu_{m, d}(t). \quad (3.69)$$

Furthermore, we will see that if we introduce continuous matter we can reproduce the entire lensing map with a positive mass singularity.

### 3.6 Complex Formulation

Before incorporating additional features into our lens, it is helpful to reframe the structure of the lensing map as a map  $\mathbb{C} \rightarrow \mathbb{C}$ , rather than a map  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ . First we will consider  $x$  as a complex number  $x_1 + ix_2$ , and likewise  $y$  and  $\eta$ . The lens equation becomes

$$\eta = \eta_1 + i\eta_2 = \left(x_1 - \frac{\partial\psi}{\partial x_1}\right) + i\left(x_2 - \frac{\partial\psi}{\partial x_2}\right) \quad (3.70)$$

Taking complex derivatives of  $\eta$  we get

$$\begin{aligned} \frac{\partial\eta}{\partial z} &= \frac{1}{2} \left( \frac{\partial\eta}{\partial x_1} - i \frac{\partial\eta}{\partial x_2} \right) = \\ &= \frac{1}{2} \left( \frac{\partial\eta_1}{\partial x_1} + i \frac{\partial\eta_2}{\partial x_1} - i \frac{\partial\eta_1}{\partial x_2} + \frac{\partial\eta_2}{\partial x_2} \right) = \frac{1}{2} \left( \frac{\partial\eta_1}{\partial x_1} + \frac{\partial\eta_2}{\partial x_2} \right), \end{aligned} \quad (3.71)$$

which is real. Here we used the equality of the mixed partials of  $\psi$  to cancel the imaginary terms. Differentiating with respect to  $\bar{z}$  gives

$$\frac{\partial \eta}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial \eta}{\partial x_1} + i \frac{\partial \eta}{\partial x_2} \right) = \frac{1}{2} \left( \frac{\partial \eta_1}{\partial x_1} - \frac{\partial \eta_2}{\partial x_2} + i \left[ \frac{\partial \eta_2}{\partial x_1} + \frac{\partial \eta_1}{\partial x_2} \right] \right). \quad (3.72)$$

Here no such cancellation occurs. We can also rewrite  $J = \det(d\eta)$  as

$$\begin{aligned} J &= \frac{\partial \eta_1}{\partial x_1} \frac{\partial \eta_2}{\partial x_2} - \frac{\partial \eta_1}{\partial x_2} \frac{\partial \eta_2}{\partial x_1} \\ &= \frac{1}{4} \frac{\partial \eta_1^2}{\partial x_1} + \frac{1}{2} \frac{\partial \eta_1}{\partial x_1} \frac{\partial \eta_2}{\partial x_2} + \frac{1}{4} \frac{\partial \eta_2^2}{\partial x_2} - \left( \frac{1}{4} \frac{\partial \eta_1^2}{\partial x_1} - \frac{1}{2} \frac{\partial \eta_1}{\partial x_1} \frac{\partial \eta_2}{\partial x_2} + \frac{1}{4} \frac{\partial \eta_2^2}{\partial x_2} \right) \\ &\quad - \left( \frac{1}{4} \frac{\partial \eta_2^2}{\partial x_1} + \frac{1}{2} \frac{\partial \eta_2}{\partial x_1} \frac{\partial \eta_1}{\partial x_2} + \frac{1}{4} \frac{\partial \eta_1^2}{\partial x_2} \right) \\ &= \left| \frac{\partial \eta}{\partial z} \right|^2 - \left| \frac{\partial \eta}{\partial \bar{z}} \right|^2. \end{aligned} \quad (3.73)$$

Here we extensively used the fact that  $\frac{\partial \eta_2}{\partial x_1} = \frac{\partial \eta_1}{\partial x_2}$ . Our critical points are located where  $J = 0$ . As we noted above  $\frac{\partial \eta}{\partial z}$  is real so the solutions to  $J = 0$  look like

$$\frac{\partial \eta}{\partial \bar{z}} = \left| \frac{\partial \eta}{\partial z} \right| e^{i\varphi}, \quad (3.74)$$

for some angle  $\varphi$ . Thus our critical curves will be curves parametrized by  $\varphi$ .

The caustics will be the images of these critical curves under  $\eta$ . Any points where the caustics aren't smooth are characterized by

$$J(x) = 0 \quad \nabla_Z(\eta) = 0. \quad (3.75)$$

Here  $Z = -\frac{\partial J}{\partial x_2} + i \frac{\partial J}{\partial x_1} = 2i \frac{\partial J}{\partial \bar{z}}$ , and

$$\nabla_Z = Z \frac{\partial}{\partial z} + \bar{Z} \frac{\partial}{\partial \bar{z}}. \quad (3.76)$$

To find the cusps on the caustics we just find the appropriate phase  $\varphi$  to solve (3.75).

### 3.7 Lensing by Negative Point Mass Singularities with Continuous Matter and Shear

Most lensing events on the scale of individual stars take place within a host galaxy. In these cases, the star itself isn't the only source of distortion. Two other non-local factors are also important.

Continuous matter is the first. The presence of evenly dispersed matter in the area of the lens can produce convergence or divergence. This enters into the dimensionless potential via a term like

$$\psi_{\text{cm}}(x) = \frac{\kappa}{2} \|x\|^2. \quad (3.77)$$

This gives us a lensing map of

$$\eta(x) = y = (1 - \kappa)x. \quad (3.78)$$

This is clearly compatible with the complex formulation. The dimensionless mass density of  $\kappa$  corresponds to a surface mass density of  $\sigma_c \pi \kappa$ .

The other factor is shear from infinity. The presence of a large mass nearby, such as a nearby galaxy, or an asymmetric distribution such as the disk of the host galaxy, can introduce a potential of the form

$$\psi_{\text{sh}}(x) = -\frac{\gamma}{2} [(x_1^2 - x_2^2) \cos 2\theta + 2x_1x_2 \sin 2\theta]. \quad (3.79)$$

The parameter  $\gamma$  defines the magnitude of the shear, and  $2\theta$  determines the preferred direction of the asymmetric mass distribution or the direction to the large mass. By the symmetry of our lens, we can assume that  $\theta = 0$ . This gives us a lensing map of

$$y = \begin{bmatrix} 1 + \gamma & 0 \\ 0 & 1 - \gamma \end{bmatrix} x. \quad (3.80)$$

In complex form this is

$$y = z + \gamma \bar{z}. \quad (3.81)$$

Now we will incorporate all of these features into a single lens. The lensing map of a negative point mass lens is given by

$$y = x \left( 1 - \frac{m}{\|x\|^2} \right). \quad (3.82)$$

In complex form this is

$$y = z - \frac{m}{\bar{z}}. \quad (3.83)$$

Adding these potentials to that of our singularity gives us the combined lens equation

$$\eta(x) = \begin{bmatrix} 1 - \kappa + \gamma & 0 \\ 0 & 1 - \kappa - \gamma \end{bmatrix} x - \frac{m}{\|x\|^2} x, \quad (3.84)$$

or

$$\eta = (1 - \kappa)z + \gamma \bar{z} - \frac{m}{\bar{z}}. \quad (3.85)$$

Our Jacobian is

$$J = \left( \gamma + \frac{m}{\bar{z}^2} \right)^2 - (1 - \kappa)^2. \quad (3.86)$$

So to look for critical points we use equation (3.74) with our  $\eta$  to give us

$$\gamma + \frac{m}{\bar{z}^2} = |1 - \kappa| e^{i\varphi}. \quad (3.87)$$

If  $\kappa = 1$ , then we are looking for points that solve

$$\frac{m}{\bar{z}^2} = \gamma. \quad (3.88)$$

Which are just two points along the  $x$  axis. So we can assume that  $\kappa \neq 1$ . It simplifies calculation to remove  $\kappa$  from the calculation by defining

$$\gamma_* = \frac{\gamma}{|1 - \kappa|} \quad m_* = \frac{m}{|1 - \kappa|} \quad \epsilon_\kappa = \text{sgn}(1 - \kappa). \quad (3.89)$$

With those substitutions we now solve

$$\gamma_* + \frac{m_*}{\bar{z}^2} = e^{i\varphi}. \quad (3.90)$$

This has solutions

$$z_{\pm}(\varphi) = \pm \sqrt{\frac{m_*}{e^{-i\varphi} - \gamma_*}}. \quad (3.91)$$

At this point the only effect that the negative sign on  $m$  has had is to rotate our critical curves by  $\pi/2$ .

$$z_{\pm}(\varphi) = \pm i \sqrt{\frac{|m_*|}{e^{-i\varphi} - \gamma_*}}. \quad (3.92)$$

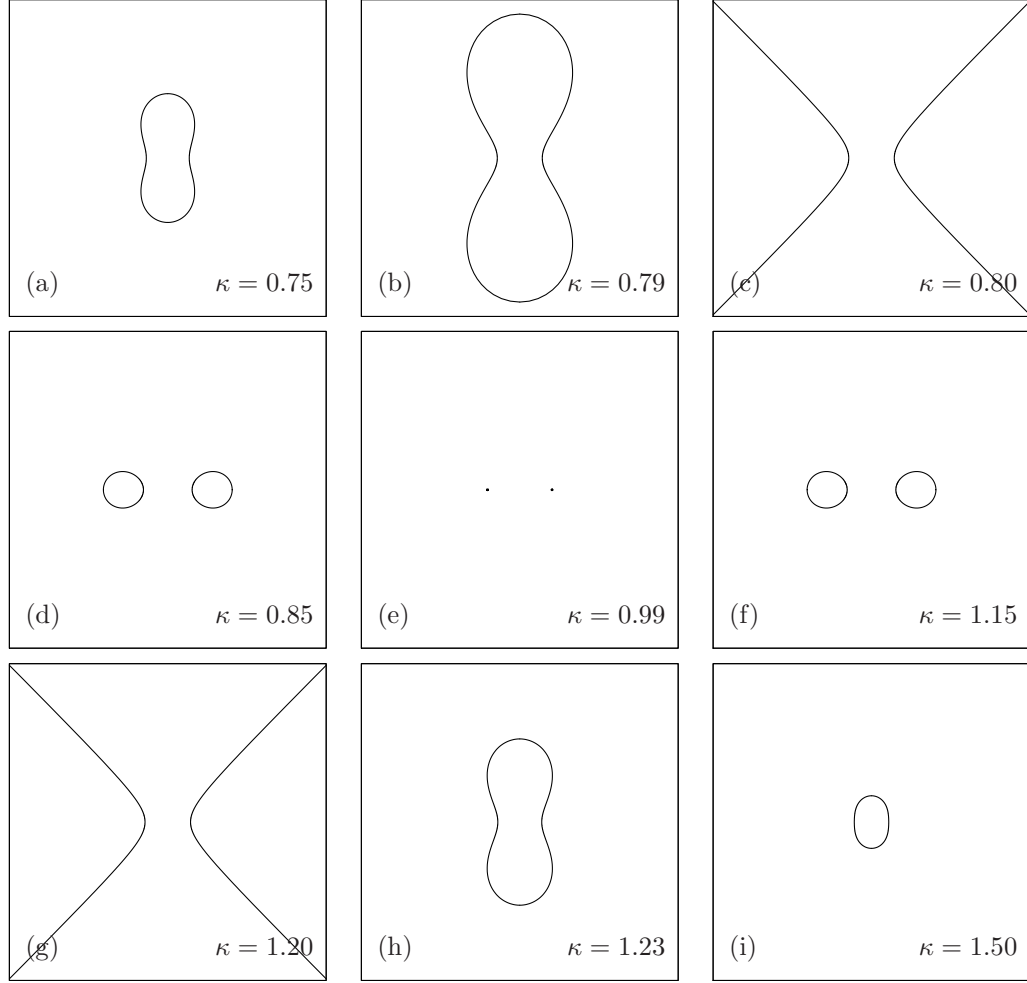
Note that these curves are independent of  $\epsilon_{\kappa}$ . For  $\gamma_* \ll 1$ , the critical curve is given by an oval with long axis in the  $x_2$  direction. As  $\gamma_*$  grows to 1, the critical curve develops a waist. For  $\gamma_* > 1$ , we have two critical curves, small loops on the  $x_1$  axis. As  $\gamma_*$  grows they shrink to the two points for  $\kappa = 1$ . For  $\gamma_*$  very close to 1, both varieties of critical curves grow to infinity. When  $\gamma_* = 1$  our critical curves degenerate into two curves asymptotic to the lines  $x_2 = \pm x_1$ . As  $\gamma_*$  passes 1, the ends of the curve open up, pass through infinity, and rejoin re-paired. See Figure 3.3 for the shapes of the critical curves for various values of shear and convergence. These curves are the same as in the positive mass case, rotated a quarter turn.

To find the cusps we plug our lensing map into (3.75). In our case we have

$$Z = -4i \left( \gamma + \frac{m}{z^2} \right) \frac{m}{\bar{z}^3} \quad (3.93)$$

and equation (3.75) is

$$0 = (1 - \kappa)Z + \left( \gamma + \frac{m}{\bar{z}^2} \right) \bar{Z}. \quad (3.94)$$



**Figure 3.3:** Critical Curves for Negative Point Mass with Continuous Matter and Shear

These pictures are shown in order of increasing  $\kappa$ , with constant  $\gamma = 0.2$ . The first four correspond to  $\epsilon_\kappa = 1$  and increasing  $\gamma_*$ . At  $\kappa = 1$  we have two points as the critical set. The last four pictures correspond to  $\epsilon_\kappa = -1$  and decreasing  $\gamma_*$ . Since the curves are independent of  $\epsilon_\kappa$ , they are the same when one replaces  $\kappa$  by  $2 - \kappa$ .

Shear	$\epsilon_\kappa = 1$		$\epsilon_\kappa = -1$	
	$\varphi_i$	$N_{\text{cusps}}$	$\varphi_i$	$N_{\text{cusps}}$
$0 \leq \gamma_*^2 < 3/4$	$\emptyset$	0	$\varphi_1, \varphi_2$	4
$3/4 \leq \gamma_*^2 < 1$	$\varphi_3, \varphi_4, \varphi_5, \varphi_6$	8	$\varphi_1, \varphi_2$	4
$1 < \gamma_*^2$	$\varphi_1, \varphi_4, \varphi_6$	6	$\varphi_2, \varphi_3, \varphi_5$	6

**Table 3.1:** Numbers of cusps on caustics for various shear and continuous matter values.

Removing the  $1 - \kappa$  as before, we get

$$0 = Z_* + \epsilon_\kappa \left( \gamma_* + \frac{m_*}{\bar{z}^2} \right) \bar{Z}_*. \quad (3.95)$$

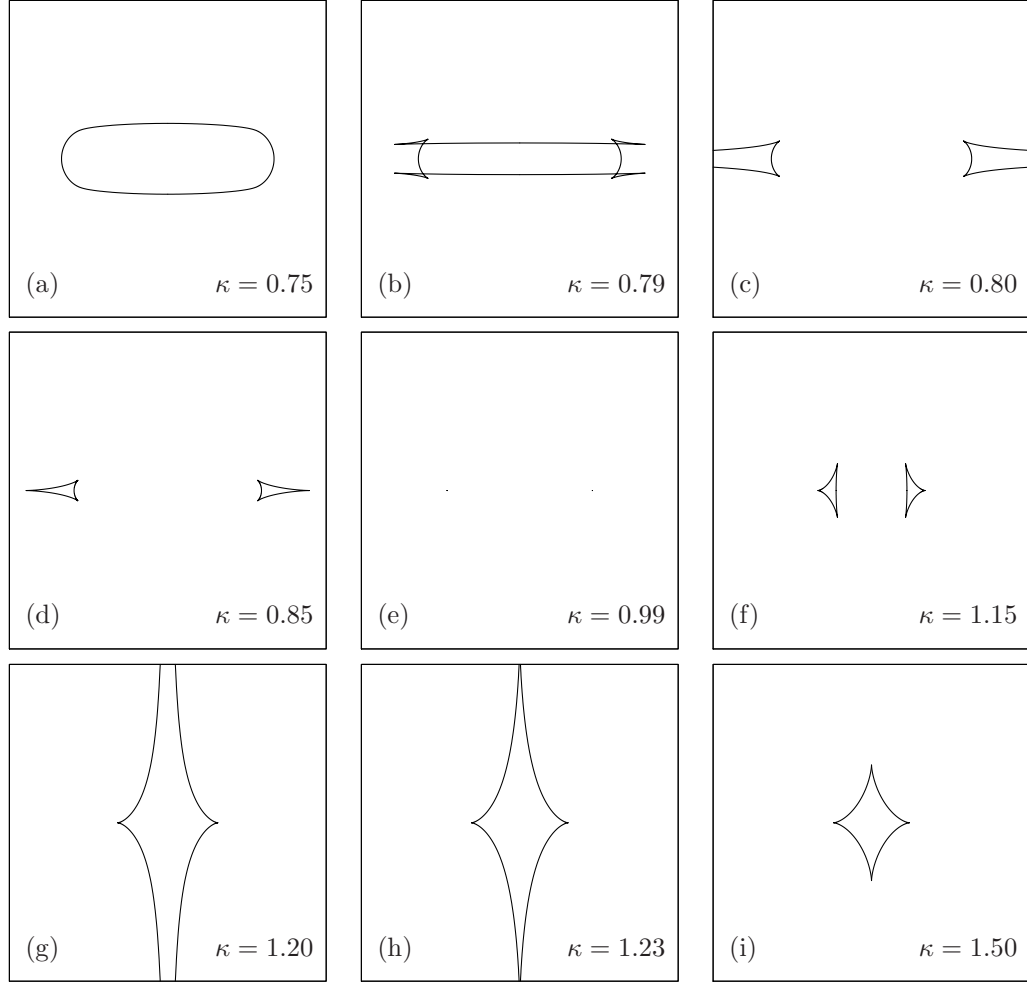
Where  $Z_*$  has the same formula as  $Z$  replacing  $\gamma$  and  $m$  with  $\gamma_*$  and  $m_*$ . In Appendix A.2 we find the possible roots of this equation to be

$$\varphi_1 = 0, \quad \varphi_2 = \pi, \quad \varphi_{3,4} = \arccos \left( \frac{3 \pm \sqrt{4\gamma_*^2 - 3}}{4\gamma_*} \right), \quad \varphi_5 = 2\pi - \varphi_3, \quad \varphi_6 = 2\pi - \varphi_4. \quad (3.96)$$

For a given value of  $\gamma_*$  and  $m_*$ , each of these is only a solution for either  $\epsilon_\kappa = 1$  or  $\epsilon_\kappa = -1$ . The numbers of cusps for various values of  $\gamma_*$  and  $\epsilon_\kappa$  are given in Table 3.1. When the number of cusps changes we expect to have higher order caustics. Looking at Figure 3.4, we see that we have four swallow tails between panels (a) and (b), we also have two elliptic umbilics between panels (d) and (f). The changes in the caustic structure between panels (b) & (d) and (f) & (h) occur at infinity.

In terms of the number of images, we always get two images outside of the caustics. Inside the caustic in panel (a) we have no images, just like in the case without shear or convergence. In panel (b) we have four images inside the swallow tails, and none inside the rectangular region. In the rest we have four images inside the caustics.

Both the critical curves and the caustics are the same as what one would get if one took the positive mass case, and switched  $\kappa$  with  $2 - \kappa$ , and rotated all the



**Figure 3.4:** Caustics for Negative Point Mass with Continuous Matter and Shear

These pictures are shown in order of increasing  $\kappa$  with constant  $\gamma = 0.2$ . The first four correspond to  $\epsilon_\kappa = 1$  and increasing  $\gamma_*$ . At  $\kappa = 1$  we have two points as the caustic set. The last four pictures correspond to  $\epsilon_\kappa = -1$  and decreasing  $\gamma_*$ .

images by a quarter turn.

# Chapter 4

## Inverse Mean Curvature Flow

This chapter lays out what we need of the weak inverse mean curvature flow as developed in [7]. We will follow their exposition closely, omitting many of the technical details.

### 4.1 Classical Formulation

Let  $N$  be a the smooth boundary of a region in the smooth Riemannian manifold  $M$ . A classical solution of the inverse mean curvature flow is a smooth family  $x : N \times [0, T] \rightarrow M$  of hypersurfaces  $N_t = x(N, t)$  satisfying the evolution equation

$$\frac{\partial x}{\partial t} = \frac{\nu}{H}. \quad (4.1)$$

Here  $\nu$  is the outward pointing normal to  $N_t$  and  $H$  is the mean curvature of  $N$ , which must be positive. We use this flow to explore the geometry near a singularity. For instance we will use the fact that under this flow the Hawking mass is non-decreasing. This result can be readily shown if we assume that the flow doesn't have any discontinuities or singularities. However, easy counterexamples illustrate that

this is overly optimistic. The simplest counterexample is given by a thin torus in  $\mathbb{R}^3$ . Such a torus has positive mean curvature approximately that of a cylinder of the small radius. The inverse mean curvature flow will tend to increase the small radius of this torus. However, if it continued without singularities or jumps, the hole in the torus would eventually shrink to the point where the area of the torus inside the hole has zero mean curvature and hence the flow couldn't be continued. Thus the classical flow is insufficient for our needs.

To remedy this problem, we will follow [7] and recast the flow first in a level set formulation and then we will move to a weak solution. This will allow the flow to jump to avoid situations where the curvature of the surface would drop to zero. In the previous example, the flow would close the interior of the torus as soon as it is favorable in terms of a certain energy functional. Even with these jumps the Hawking mass of our surface is still non-decreasing.

## 4.2 Weak Formulation

First we establish some notation. Let  $(M, g)$  be the ambient manifold. Let  $h$  be the induced metric on  $N$ . Let  $A_{ij} = \langle \nabla_{e_i} \nu, e_j \rangle$  be the second fundamental form of  $N$ . Then  $H$  is the trace of  $A$  with  $h$ , and  $\vec{H} = -\nu H$  is the mean curvature vector. Let  $E$  be the open region bounded by  $N$ .

The first step toward the weak formulation is a level set formulation. We assume that the flow is given by the level sets of a function  $u : M \rightarrow \mathbb{R}$ . This  $u$  is related to our previous data by

$$E_t := \{x \mid u(x) < t\}, \quad N_t := \partial E_t. \quad (4.2)$$

We will also need the following sets

$$E_t^+ := \text{int} \{x \mid u(x) \geq t\}, \quad N_t^+ := \partial E_t^+ \quad (4.3)$$

Where  $\nabla u \neq 0$ ,  $E_t = E_t^+$  and  $N_t = N_t^+$ .

Anywhere that  $u$  is smooth and  $\nabla u \neq 0$ , then we have a foliation by smooth surfaces  $N_t$ , with normal vector  $\nu = \nabla u / |\nabla u|$ . The mean curvature of these surfaces is given by  $\text{div}_N(\nu)$  and the flow velocity is given by  $1/|\nabla u|$ , so equation (4.1) becomes

$$\text{div}_M \left( \frac{\nabla u}{|\nabla u|} \right) = |\nabla u|. \quad (4.4)$$

This equation is degenerate elliptic. To remedy this we introduce the functional  $J_u^K(v)$ :

$$J_u(v) = J_u^K(v) = \int_K |\nabla v| + v |\nabla u|. \quad (4.5)$$

Where  $K$  is a compact set in  $M$ . If we take the Euler-Lagrange equation of this functional, and replace  $v$  with  $u$ , we get back equation (4.4). For each  $u$  we get a different  $J_u$ . Hence what we want is a function  $u$  which minimizes its own  $J_u$ .

**Definition 4.2.1.** Let  $u$  be a locally Lipschitz function on the open set  $\Omega$  in  $M$ .

Then  $u$  is a *weak (sub-, super-) solution* of equation (4.4) on  $\Omega$  exactly when

$$J_u^K(u) \leq J_u^K(v) \quad (4.6)$$

for all locally Lipschitz functions  $v$  ( $\leq u, \geq u$ ) which only differ from  $u$  inside a compact set  $K$  contained in  $\Omega$ .

It is worth noting that  $J_u(\min(v, w)) + J_u(\max(v, w)) = J_u(v) + J_u(w)$ . To see this, construct  $K_v = \{x \in K \mid v(x) < w(x)\}$ , and divide the integrals on the left into

integrals over  $K_v$  and  $K \setminus K_v$ . Then regroup then and recombine to get the right hand side. This tells us that if  $u$  is both a weak supersolution and subsolution, then it is a weak solution.

We will also need a related functional of the level sets.

**Definition 4.2.2.** If  $F$  is a set of locally finite perimeter, and  $\partial^* F$  is its reduced boundary, then we define

$$J_u(F) = J_u^K(F) = |\partial^* F \cap K| - \int_{F \cap K} |\nabla u|. \quad (4.7)$$

For any locally Lipschitz function  $u$  and compact  $K$  contained in  $A$ . We say that  $E$  minimizes  $J_u$  in  $A$  (on the inside, outside) if

$$J_u^K(E) \leq J_u(F) \quad (4.8)$$

for all  $F$  that differs from  $E$  in some compact  $K$  in  $A$  (with  $E \subseteq F$ ,  $E \supseteq F$ .) A similar argument tells us that if  $E$  minimizes  $J_u$  exactly when it minimizes  $J_u$  on the inside and outside.

These two formulations are equivalent.

**Lemma 4.2.3.** *Let  $u$  be a locally Lipschitz function in the open set  $\Omega$ . Then  $u$  is a weak (sub-, super-) solution of equation (4.4) exactly when for each  $t$ ,  $E_t = \{u < t\}$  minimizes  $J_u$  in  $\Omega$  (on the inside, outside).*

*Proof.* Lemma 1.1 in [7]. □

We now define the initial value problem. Let  $E_0$  be an open set with  $C^1$  boundary. We say that  $u \in C_{\text{loc}}^{0,1}$  and the associated  $E_t$  for  $t > 0$  is a *weak solution* of (4.4) with

initial condition  $E_0$  if either

$$E_0 = \{u < 0\} \text{ and } u \text{ minimizes } J_u \text{ on } M \setminus E_0$$

$$\text{or} \tag{4.9}$$

$$E_t = \{u < t\} \text{ minimizes } J_u \text{ in } M \setminus E_0 \text{ for } t > 0.$$

These two conditions are equivalent by Lemma 1.2 in [7]. Showing the regularity of  $N_t$  and  $N_t^+$  is nontrivial, but we won't reproduce it here.

**Theorem 4.2.4.** *Let  $n < 8$ . Let  $U$  be an open set in a domain  $\Omega$ . Let  $f$  be a bounded measurable function on  $\Omega$ . Consider the functional*

$$|\partial F| + \int_F f \tag{4.10}$$

*on sets containing  $U$  and compactly contained in  $\Omega$ . Suppose  $E$  minimizes this functional.*

1. *If  $\partial U$  is  $C^1$ , then  $\partial E$  is a  $C^1$  submanifold of  $\Omega$ .*
2. *If  $\partial U$  is  $C^{1,\alpha}$ ,  $0 < \alpha \leq 1/2$ , then  $\partial E$  is a  $C^{1,\alpha}$  submanifold of  $\Omega$ . The  $C^{1,\alpha}$  estimates depend only on the distance to  $\partial\Omega$ ,  $\text{ess sup } |f|$ ,  $C^{1,\alpha}$  bound for  $\partial U$ , and  $C^1$  bounds on the metric in  $\Omega$ .*
3. *If  $\partial U$  is  $C^2$  and  $f = 0$ , then  $\partial E$  is  $C^{1,1}$ , and  $C^\infty$  where it doesn't touch  $U$ .*

Our initial value formulation falls into this category of problem. So our  $N_t$ 's and  $N_t^+$ 's are  $C^{1,\alpha}$ . Furthermore

$$\lim_{s \rightarrow t^-} N_s = N_t \quad \lim_{s \rightarrow t^+} N_s = N_t^+ \tag{4.11}$$

in local  $C^{1,\beta}$  convergence  $0 < \beta \leq \alpha$ .

The locations where  $N_t \neq N_t^+$  correspond to the jumps discussed earlier. To examine these areas we need to introduce minimizing hulls.

**Definition 4.2.5.** Let  $\Omega$  be an open set. We call  $E$  a *minimizing hull* if

$$|\partial^* E \cap K| \leq |\partial^* F \cap K| \quad (4.12)$$

for any  $F$  containing  $E$  with  $F \setminus E$  in  $K$  a compact set in  $\Omega$ . We say  $E$  *strictly minimizes* if equality implies that  $E$  and  $F$  agree in  $\Omega$  up to measure zero.

The intersection of (strictly) minimizing hulls is a (strictly) minimizing hull. So, given a set  $E$  we can intersect all of the strictly minimizing hulls which contain  $E$ . This gives, up to measure zero, a unique set  $E'$  that we will call *the strictly minimizing hull of  $E$* . Since  $E'$  is strictly minimizing,  $E'' = E'$ .

Solutions to the initial value problem given by equation (4.9), have level sets that are minimizing hulls as follows.

**Lemma 4.2.6.** *Suppose that  $u$  is a solution to (4.9). Assume that  $M$  has no compact components. Then:*

- For  $t > 0$ ,  $E_t$  is a minimizing hull in  $M$ .
- For  $t > 0$ ,  $E_t^+$  is a strictly minimizing hull in  $M$ .
- For  $t > 0$ ,  $E_t' = E_t^+$  if  $E_t^+$  is precompact.
- For  $t > 0$ ,  $|\partial E_t| = |\partial E_t^+|$ , provided that  $E_t^+$  is precompact.
- Exactly when  $E_0$  is a minimizing hull  $|\partial E_0| = |\partial E_0^+|$

This is “Minimizing Hull Property (1.4)” in [7]. These minimizing properties characterize how the weak flow differs from the classical flow. The classical flow runs into trouble when the mean curvature changes sign. If the mean curvature is

negative on a part of  $N_t$ , we could decrease the area of  $N_t$  by flowing that patch out. So the weak flow avoids these areas by making sure that  $N_t^+$  is always the outermost surface with its area. This sometimes necessitates jumping.

The existence and uniqueness of solutions to equation (4.4) with initial data are beyond our scope. However, for completeness, here is the existence and uniqueness theorem (3.1) from [7].

**Theorem 4.2.7.** *Let  $M$  be a complete, connected Riemannian  $n$ -manifold without boundary. Suppose there exists a proper, locally Lipschitz, weak subsolution of (4.9) with a precompact initial condition.*

*Then for any nonempty, precompact, smooth open set  $E_0$  in  $M$ , there exists a proper, locally Lipschitz solution  $u$  of (4.9) with initial condition  $E_0$ , which is unique in  $M \setminus E_0$ . Furthermore, the gradient of  $u$  satisfies the estimate*

$$|\nabla u(x)| \leq \sup_{\partial E_0 \cap B_r(x)} H_+ + \frac{C(n)}{r}, \quad \text{a.e. } x \in M \setminus E_0, \quad (4.13)$$

*for each  $0 < r \leq \sigma(x)$ .*

The function  $\sigma(x)$  depends on the Ricci curvature of  $M$  near  $E_0$ , but is always positive. More importantly, the requirement for a subsolution is satisfied by any asymptotically flat manifold. A function like  $\ln(R)$  in the asymptotic end will suffice.

### 4.3 Useful Properties of Weak IMCF

Now that we have outlined the flow, we will describe some useful properties.

**Lemma 4.3.1.** *Let  $(E_t)_{t>0}$  solve (4.9) with initial condition  $E_0$ . As long as  $E_t$  remains precompact, we have the following:*

- $e^{-t} |\partial E_t|$  is constant for  $t > 0$ .
- If  $E_0$  is its own minimizing hull then  $|\partial E_t| = e^t |\partial E_0|$ .

*Proof.* Since each  $E_t$  minimizes the same functional, they must all have the same value for  $J_u(E_t)$ . Applying the co-area formula to the integral in  $J_u$  gives

$$J_u(E_t) = |\partial E_t| - \int_0^t \frac{1}{|\nabla u|} \int_{\partial E_s} |\nabla u| \, dA \, ds \quad (4.14)$$

$$= |\partial E_t| - \int_0^t |\partial E_s| \, ds. \quad (4.15)$$

Which has solutions of the form  $Ce^t$ . By the minimizing hull properties, we could replace  $\partial E_t$  with  $\partial E_t^+$  for  $t > 0$ . Since  $\partial E_0^+$  is the limit of  $\partial E_s$  as  $s \searrow 0$ ,  $C = |\partial E_0^+|$ . If  $E_0$  is its own minimizing hull then  $|\partial E_0| = |\partial E_0^+|$ , and  $C = |\partial E_0|$ .  $\square$

The next two lemmas tell us that when the classical solution exists, it agrees with the weak solution.

**Lemma 4.3.2.** *Let  $(N_t)_{c \leq t < d}$  be a smooth family of surfaces of positive mean curvature that solve (4.1) classically. Let  $u = t$  on  $N_t$ ,  $u < c$  inside  $N_c$ , and  $E_t = \{u < t\}$ . Then for  $c \leq t < d$ ,  $E_t$  minimizes  $J_u$  in  $E_d \setminus E_c$ .*

**Lemma 4.3.3.** *Let  $E_0$  be a precompact open set in  $M$  such that  $\partial E_0$  is smooth with  $H > 0$  and  $E_0 = E'_0$ . Then any unique solution  $(E_t)_{0 < t < \infty}$  of (4.9) with initial condition  $E_0$  coincides with the unique smooth classical solution for a short time, provided  $E_t$  remains precompact for a short time.*

The authors of [7] point out that the stopping point for these theorems will be when either  $E_t$  is no longer a minimizing hull, the mean curvature goes to zero, or the second fundamental form is unbounded.

The proof of existence and uniqueness of the weak flow are beyond the scope of this thesis. However, the method is as follows. First they assume the existence of a subsolution  $v$ . Then they solve the regularized problem

$$\begin{aligned}
E^\epsilon u^\epsilon &:= \operatorname{div} \left( \frac{\nabla u^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} \right) - \sqrt{|\nabla u^\epsilon|^2 + \epsilon^2} = 0 && \text{in } \Omega_L \\
u^\epsilon &= 0 && \text{on } \partial E_0 \\
u^\epsilon &= L - 2 && \text{on } \partial F_L.
\end{aligned} \tag{4.16}$$

Where  $F_L = \{v < L\}$ ,  $\epsilon$  is small, and  $L$  is large, but bounded in size by a function of  $\epsilon$ . Then they take the limit as  $L \rightarrow \infty$ ,  $\epsilon \rightarrow 0$ . Some estimates on  $|\nabla u|$  and  $H$  guarantee that the solution passes to the limit.

The authors also proved that in the asymptotic regime, the Hawking mass of the level sets of the flow converges to the ADM mass of the manifold

**Theorem 4.3.4.** *Assume that  $M$  is asymptotically flat and let  $(E_t)_{t \geq t_0}$  be a family of precompact sets weakly solving (4.1) in  $M$ . Then*

$$\lim_{t \rightarrow \infty} m_H(N_t) \leq m_{ADM}(M). \tag{4.17}$$

*Proof.* They show that  $N_t$  must approach coordinate spheres in the asymptotic regime. This is their Lemma 7.4. □

## 4.4 Geroch Monotonicity

The Geroch Monotonicity formula says that the Hawking mass is nondecreasing under the inverse mean curvature flow. The original use of this was to propagate the mass of a surface out to infinity to compare to the ADM mass using IMCF. However,

it also provides bounds on integrals of  $H$  and area near the surface. We will first show how it arises in the smooth case, and then extend it over the jumps in the weak flow.

In the smooth case, we simply recall that the Hawking mass is given by

$$m_H = \sqrt{\frac{|N|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_N H^2 \right). \quad (4.18)$$

The first variation of area is given by

$$\frac{d}{dt} dA = H\eta dA. \quad (4.19)$$

Thus under smooth IMCF we have  $\frac{d}{dt} dA = dA$ . The variation of  $H$  is given by

$$\frac{dH}{dt} = \Delta(-\eta) - |A|^2 \eta - \text{Rc}(\nu, \nu)\eta. \quad (4.20)$$

Thus if we look at the integral  $\int H^2 dA$  under IMCF we get

$$\frac{d}{dt} \int H^2 dA = \int H^2 - 2 \frac{|\nabla_N H|^2}{H^2} + 2|A|^2 - 2 \text{Rc}(\nu, \nu) dA. \quad (4.21)$$

The Gauss equation contracts to give

$$K = K_{12} + \lambda_1 \lambda_2 = \frac{R}{2} - \text{Rc}(\nu, \nu) + \frac{1}{2}(H^2 - |A|^2) \quad (4.22)$$

Here  $K$  and  $R$  are the scalar curvatures of  $N$  and  $M$  respectively,  $K_{12}$  is the sectional curvature of  $M$  in the plane tangent to  $N$ , and  $\lambda_i$  are the principal curvatures of  $N$ .

Using this equation to cancel the  $\text{Rc}$  term gives us the following equation

$$\frac{d}{dt} \int_N H^2 = \int_N 2K - 2 \frac{|\nabla_N H|^2}{H^2} - |A|^2 - R \quad (4.23)$$

$$= \int_N 4K - R - 2 \frac{|\nabla_N H|^2}{H^2} - \frac{1}{2}(\lambda_1 + \lambda_2)^2 - \frac{1}{2}(\lambda_1 - \lambda_2)^2. \quad (4.24)$$

If  $R > 0$ ,

$$\frac{d}{dt} \int_N H^2 = 4\pi\chi(N_t) - \frac{1}{2} \int_N H^2 - \int_N 2 \frac{|\nabla_N H|^2}{H^2} + \frac{1}{2} (\lambda_1 - \lambda_2)^2. \quad (4.25)$$

If  $N_t$  is connected,

$$\frac{d}{dt} \int_N H^2 \leq 8\pi \left( 1 - \frac{1}{16\pi} \int_N H^2 \right). \quad (4.26)$$

In addition recall that  $|N_t| = |N_0| e^t$  in the smooth case. Thus we get that

$$\sqrt{16\pi} \frac{d}{dt} m_H = \frac{d}{dt} \left[ e^{t/2} \left( 1 - \frac{1}{16\pi} \int_N H^2 \right) \right] \quad (4.27)$$

$$\geq \left[ \frac{1}{2} e^{t/2} \left( 1 - \frac{1}{16\pi} \int_N H^2 \right) - e^{t/2} \frac{8\pi}{16\pi} \left( 1 - \frac{1}{16\pi} \int_N H^2 \right) \right] = 0. \quad (4.28)$$

Hence the Hawking mass is nondecreasing.

To cover the gap, we simply note that at the jumps, the new surface  $E'_t = E_t^+$  is a minimizing hull for  $E_t$ . That means that where their boundaries differ,  $\partial E'_t$  must have zero mean curvature (else a variation could decrease its area keeping it outside of  $E_t$ , contradicting its minimizing property.) Hence

$$\int_{\partial E_t} H^2 \geq \int_{\partial E'_t} H^2. \quad (4.29)$$

Thus since  $|\partial E'_t| = |\partial E_t|$  for  $t > 0$ , we see that the, even at jumps, the Hawking mass can't decrease. This doesn't cover the possibility of dense jumping, or similar pathological behavior, but extending the Geroch formula to those cases requires using the elliptic regularization.

The statement of Geroch Monotonicity given in [7] for one boundary component is as follows

**Theorem 4.4.1.** *Let  $M$  be a complete 3-manifold,  $E_0$  a precompact open set with  $C^1$  boundary satisfying  $\int_{\partial E_0} |A|^2 < \infty$ , and  $(E_t)_{t>0}$  a solution to (4.9) with initial condition  $E_0$ . If  $E_0$  is a minimizing hull then*

$$m_H(N_s) \geq m_H(N_r) + \frac{1}{(16\pi)^{3/2}} \int_r^s \left[ 16\pi - 8\pi\chi(N_t) + \int_{N_t} (2|D \log H|^2 + (\lambda_1 - \lambda_2)^2 + R) \right] dt \quad (4.30)$$

for  $0 \leq r \leq s$  provided  $E_s$  is precompact.

# Chapter 5

## Negative Point Mass Singularity Results

### 5.1 Negative Point Mass Singularities and IMCF

Although we do not need this fact, it is interesting to note that near a negative point mass singularity, we can define the inverse mean curvature flow. Even though there is no initial surface to start from we can take a limit of solutions to IMCF for starting surfaces that converge to  $p$ . This actually defines a unique solution  $u$  to the weak inverse mean curvature flow. This was shown in recent work by Jeffrey Streets [10].

**Theorem 5.1.1** ([10]). *Let  $M^3$  be an asymptotically flat manifold with finitely many singularities at  $\{p_i\}$ . Then there is a unique solution to (4.4) on  $M \setminus \{p_i\}$ .*

In this case since the area of the level sets is exponential in time, and surfaces near the singularities have vanishing area. Thus this flow only reaches the singularity at time  $-\infty$ .

Streets also showed that these surfaces were the best possible surfaces, in terms of the Hawking mass, as in the following theorem.

**Theorem 5.1.2** ([10]). *Let  $S_t$  be the family of hypersurfaces defining the solution to IMCF above. Let  $P_t$  be any other family of hypersurfaces approaching the singularity. Then,*

$$\lim_{t \rightarrow -\infty} m_H(P_t) \leq \lim_{t \rightarrow -\infty} m_H(S_t). \quad (5.1)$$

We can also extend Geroch Monotonicity down to  $t = 0$  in the case where our initial surface has negative Hawking mass.

**Lemma 5.1.3.** *Let  $\Sigma$  be a surface in an asymptotically flat 3 manifold. Let  $\Sigma'$  be the boundary of the minimizing hull of  $\Sigma$ . Let  $\Sigma$  or  $\Sigma'$  have negative Hawking mass. Then*

$$m_H(\Sigma) \leq m_H(\Sigma'). \quad (5.2)$$

*Proof.* If  $\Sigma'$  has nonnegative Hawking mass then  $m_H(\Sigma') \geq 0 \geq m_H(\Sigma)$  and we are done. Thus we can assume that  $\Sigma'$  has negative Hawking mass. Since  $\Sigma'$  has negative Hawking mass, it must intersect  $\Sigma$  on a set of positive measure. Otherwise,  $\Sigma'$  would be a minimal surface, with Hawking mass  $\sqrt{\frac{|\Sigma'|}{16\pi}} > 0$ . We define the following sets:

$$\Sigma_0 = \Sigma' \cap \Sigma \quad \Sigma_+ = \Sigma' \setminus \Sigma_0 \quad \Sigma_- = \Sigma \setminus \Sigma_0 \quad (5.3)$$

Recalling that  $|\Sigma_+| \leq |\Sigma_-|$  by the minimization property, and that  $H = 0$  on  $\Sigma_+$ , we observe the following:

$$0 > m_H(\Sigma') = \frac{\sqrt{|\Sigma_0| + |\Sigma_+|}}{(16\pi)^{3/2}} \left( 16\pi - \int_{\Sigma_0} H^2 \right) \quad (5.4)$$

$$\geq \frac{\sqrt{|\Sigma_0| + |\Sigma_-|}}{(16\pi)^{3/2}} \left( 16\pi - \int_{\Sigma_0} H^2 \right) \quad (5.5)$$

$$\geq \frac{\sqrt{|\Sigma_0| + |\Sigma_-|}}{(16\pi)^{3/2}} \left( 16\pi - \int_{\Sigma_0} H^2 - \int_{\Sigma_-} H^2 \right) = m_H(\Sigma). \quad (5.6)$$

□

With this lemma and Geroch Monotonicity we can prove the following lemma.

**Lemma 5.1.4.** *Let  $(M, g)$  be an asymptotically flat manifold with ADM mass  $m$ , nonnegative scalar curvature and a single regular negative point mass singularity  $p$ . Let  $\{\Sigma_i\}$  be a smooth family of surfaces converging to  $p$ , which eventually have negative Hawking mass. Then for sufficiently large  $i$ ,*

$$m_H(\Sigma_i) \leq m. \quad (5.7)$$

*Proof.* Since for large enough  $i$ ,  $\Sigma_i$  has non-positive Hawking mass, we can apply Lemma 5.1.3 to show that  $\Sigma'_i$  must have larger Hawking mass. From this surface, we start Inverse Mean Curvature Flow. Theorem 4.4.1 tells us that the Hawking mass of the surfaces  $N_t$  defined by IMCF starting with  $\Sigma'_i$  only increase. Theorem 4.3.4 tells us that the increasing limit of the Hawking masses these surfaces is less than the ADM mass. Thus the Hawking mass of the starting surface was also less than the ADM mass.  $\square$

Now we relate the limit of the Hawking masses to the regular mass.

**Lemma 5.1.5.** *Let  $(M, g)$  be an asymptotically flat manifold with nonnegative scalar curvature and a single regular negative point mass singularity  $p$ . Then there is a smooth family of surfaces  $\{\Sigma_i\}$  converging to  $p$  such that*

$$\lim_{i \rightarrow \infty} m_H(\Sigma_i) = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma} \overline{\nu}(\overline{\varphi})^{4/3} dA \right)^{3/2} = m_R(p). \quad (5.8)$$

*Proof.* The Hawking mass of a surface  $\Sigma_i$  is given by

$$m_H(\Sigma_i) = \sqrt{\frac{|\Sigma_i|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_i} H^2 dA \right). \quad (5.9)$$

Since the areas of the surfaces are converging to zero we have

$$\lim_{i \rightarrow \infty} m_H(\Sigma_i) = - \lim_{i \rightarrow \infty} \frac{\sqrt{|\Sigma_i|}}{(16\pi)^{3/2}} \int_{\Sigma_i} H^2 dA. \quad (5.10)$$

By the Hölder inequality this is bounded as follows

$$-\frac{\sqrt{|\Sigma_i|}}{(16\pi)^{3/2}} \int_{\Sigma_i} H^2 dA \leq -\frac{1}{(16\pi)^{3/2}} \left( \int_{\Sigma_i} H^{4/3} dA \right)^{3/2}. \quad (5.11)$$

Switching to the resolution space, we use the formula

$$H = \overline{\varphi}^{-2} \overline{H} + 4\overline{\varphi}^{-3} \overline{\nu}(\overline{\varphi}). \quad (5.12)$$

Putting this into the previous equation we get

$$\int_{\Sigma_i} H^{4/3} dA = \int_{\Sigma_i} (\overline{\varphi}^{-2} \overline{H} + 4\overline{\varphi}^{-3} \overline{\nu}(\overline{\varphi}))^{4/3} \varphi^4 \overline{dA} \quad (5.13)$$

$$= \int_{\overline{\Sigma}_i} (\overline{\varphi} \overline{H} + 4\overline{\nu}(\overline{\varphi}))^{4/3} \overline{dA}. \quad (5.14)$$

Since  $\overline{\varphi}$  is zero on  $\overline{\Sigma}$  and  $\overline{H}$  is bounded, the first term goes to zero. The second term converges since the family of surfaces  $\{\Sigma_i\}$  are converging smoothly.

$$\lim_{i \rightarrow \infty} \int_{\overline{\Sigma}_i} (\overline{\varphi} \overline{H} + 4\overline{\nu}(\overline{\varphi}))^{4/3} \overline{dA} = 4^{4/3} \int_{\overline{\Sigma}} \overline{\nu}(\overline{\varphi})^{4/3} \overline{dA}. \quad (5.15)$$

Combining all of these equations we have

$$\lim_{i \rightarrow \infty} m_H(\Sigma_i) \leq -\frac{1}{4} \left( \frac{1}{\pi} \int_{\overline{\Sigma}} \overline{\nu}(\overline{\varphi})^{4/3} \overline{dA} \right)^{3/2} = m_R(p). \quad (5.16)$$

To see when this estimate is sharp, we look at inequality (5.11) since that is the only inequality is our estimate. In the limit, this inequality is an equality exactly when the ratio of the maximum and minimum values of  $H$  approaches 1. We choose a

resolution such that  $\overline{\nu}(\overline{\varphi}) = 1$  on the boundary. We also choose a family of surfaces  $\Sigma_i$  given by level sets of  $\overline{\varphi}$ . Then if we look at the ratio

$$\lim_{\varphi \rightarrow 0} \frac{H_{\min}}{H_{\max}} = \lim_{\varphi \rightarrow 0} \frac{\overline{\varphi} \overline{H}_{\min} + 4\overline{\nu}(\overline{\varphi})}{\overline{\varphi} \overline{H}_{\max} + 4\overline{\nu}(\overline{\varphi})}, \quad (5.17)$$

and remember that  $\overline{H}$  is bounded, we see that the  $\overline{\nu}(\overline{\varphi})$  terms dominate, and as  $\overline{\varphi} \rightarrow 0$ , this ratio approaches 1. Thus with this resolution and this family of surfaces, inequality (5.16) will turn to an equality.  $\square$

With these results we can prove the following theorem

**Theorem 5.1.6.** *Let  $(M, g)$  be an asymptotically flat manifold with nonnegative scalar curvature and a single regular negative point mass singularity  $p$ . Then the ADM mass of  $M$  is at least the mass of  $p$ .*

*Proof.* First consider the case when  $p$  can be enclosed by a surface,  $\Sigma$ , with nonnegative Hawking mass. The minimizing hull of a surface with nonnegative Hawking mass has nonnegative Hawking mass. Thus we can run IMCF from  $\Sigma'$ , and the AMD mass of  $M$  is at least  $m_H(\Sigma') \geq 0$ . However, the regular mass of  $p$  is always nonpositive so in this case we are done.

Now assume that  $p$  cannot be enclosed by a surface with nonnegative Hawking mass. By Lemma 5.1.4 we know that the ADM mass is greater than the Hawking masses of any sequence of surface converging to  $p$  which have negative Hawking mass. By Lemma 5.1.5 we know that there is a family of surfaces converging to  $p$  which have the mass of  $p$  as the limit of their Hawking mass, hence the ADM mass is greater than their Hawking masses which limit to the regular mass.  $\square$

This can be extended to general negative point mass singularities. However, first we need to consider the effect of multiplication by a harmonic conformal factor on the ADM mass of a manifold.

**Lemma 5.1.7.** *Let  $(M^3, g)$  be an asymptotically flat manifold. Let  $\varphi$  be a harmonic function with respect to  $g$  with asymptotic expansion*

$$\varphi = 1 + \frac{C}{|x|_g} + \mathcal{O}\left(\frac{1}{|x|_g^2}\right). \quad (5.18)$$

*Then, if the ADM mass of  $(M^3, g)$  is  $m$ , the ADM mass of  $(M^3, \varphi^4 g)$  is  $m + 2C$ .*

*Proof.* This is a direct calculation. We write  $g^\varphi = g\varphi^4$ , and calculate, only keeping the terms of lowest order in  $1/|x|$  since we are taking limits as  $|x| \rightarrow \infty$ .

$$m_\varphi = \lim_{|x| \rightarrow \infty} \frac{1}{16\pi} \int_{S^\delta} (g_{ij,i}^\varphi - g_{ii,j}^\varphi) n^j dA \quad (5.19)$$

$$= \lim_{|x| \rightarrow \infty} \frac{\varphi^4}{16\pi} \int_{S^\delta} (g_{ij,i} - g_{ii,j}) n^j dA + \lim_{|x| \rightarrow \infty} \frac{\varphi^3}{4\pi} \int_{S^\delta} (\delta_{ij}\varphi_i - \delta_{ii}\varphi_j) n^j dA \quad (5.20)$$

$$= \lim_{|x| \rightarrow \infty} \varphi^4 m + \lim_{|x| \rightarrow \infty} \varphi^3 \lim_{|x| \rightarrow \infty} \frac{1}{4\pi} \int_{S^\delta} (\varphi_j - 3\varphi_j) n^j dA \quad (5.21)$$

$$= m + \lim_{|x| \rightarrow \infty} \frac{1}{2\pi} \int_{S^\delta} \varphi_j n^j dA \quad (5.22)$$

$$= m + \lim_{|x| \rightarrow \infty} \frac{1}{2\pi} \int_{S^\delta} \langle \nabla \varphi, \nu \rangle dA \quad (5.23)$$

$$= m + 2C. \quad (5.24)$$

□

Using this we can now extend Theorem 5.1.6 to a general singularity.

**Theorem 5.1.8.** *Let  $(M, g)$  be an asymptotically flat manifold with nonnegative scalar curvature and a single negative point mass singularity  $p$ . Then  $m$ , the ADM mass of  $M$ , is at least the mass of  $p$ .*

*Proof.* If the capacity of  $p$  is nonzero, then the statement is trivial. Thus we assume the capacity of  $p$  is zero. Using the terminology of Definition 2.3.4, Theorem 5.1.6 tells us that the ADM mass of  $(M, h_i^4 g)$  is at least the mass of the regular singularity at  $\Sigma_i = p_i$ . Each  $h_i$  is defined by the equations

$$\Delta h_i = 0 \tag{5.25}$$

$$\lim_{x \rightarrow \infty} h_i = 1 \tag{5.26}$$

$$h_i = 0 \text{ on } \Sigma_i. \tag{5.27}$$

Thus it has asymptotic expansion

$$h_i = 1 - \frac{C_i}{|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{5.28}$$

Where  $C_i$  is the capacity of  $\Sigma_i$ . Thus, the ADM mass,  $m_i$ , of  $(M, h_i^4 g)$  is given by  $m - 2C_i$ . Now we know that  $m_i \geq m_R(p_i)$ . Taking  $\overline{\lim}$  of both sides gives us

$$\overline{\lim}_{i \rightarrow \infty} m_i \geq \overline{\lim}_{i \rightarrow \infty} m_R(p_i) \tag{5.29}$$

Since  $C_i$  is going to zero, the left hand side is simply  $m$ , and so has no dependence on which  $\{\Sigma_i\}$  we chose in our mass calculation. Thus we get

$$m \geq \sup_{\{\Sigma_i\}} \overline{\lim}_{i \rightarrow \infty} m_R(p_i). \tag{5.30}$$

as desired. □

## 5.2 Capacity Theorem

Perhaps the most important new results of this thesis are Theorems 5.2.4 and 5.2.5 which relate the capacity of a point to the Hawking masses of surfaces near that point.

The capacity of a surface provides a measure of its size as seen from infinity. We extend the definition of the capacity of surface to the capacity of a negative point mass singularity. We then show that if a NPMS has non-zero capacity the Hawking mass of any family of surfaces converging to it must go to negative infinity. First the definition of capacity:

**Definition 5.2.1.** Let  $\Sigma$  be surface in an asymptotically flat manifold  $M$ . Define the *capacity* of  $\Sigma$  by

$$C(\Sigma) = \inf \left\{ \int_M \|\nabla \varphi\|^2 dV \mid \varphi(\Sigma) = 1, \varphi(\infty) = 0 \right\}. \quad (5.31)$$

It is worth noting that if  $\Sigma$  and  $\Sigma'$  are two surfaces in  $M$  so that  $\Sigma$  divides  $M$  into two components, one containing infinity and the other containing  $\Sigma'$ , then

$$C(\Sigma') \leq C(\Sigma) \quad (5.32)$$

since the infimum is over a larger set of functions.

We will next define the capacity of a singular point. The natural definition is the one we want.

**Definition 5.2.2.** Let  $p$  be singular point in an asymptotically flat manifold  $M$ . Chose a sequence of surfaces  $\Sigma_i$  of decreasing diameter enclosing  $p$ . Then define the *capacity* of  $p$  by the limit of the capacities of  $\Sigma_i$ .

Before using this definition we have to show that it is unique.

**Lemma 5.2.3.** *Let  $\Sigma_i$  and  $\tilde{\Sigma}_i$  be two sequences of surfaces approaching the point  $p$ . If  $\lim C(\Sigma_i) = K$ ,  $\lim C(\tilde{\Sigma}_i) = K$ . Hence  $C(p)$  is well defined.*

*Proof.* Since the  $\Sigma_i$  are going to  $p$ , for any given  $\tilde{\Sigma}_{i_0}$ , we can choose  $i_0$  such that for all  $i > i_0$ ,  $\Sigma_i$  is contained within  $\tilde{\Sigma}_{i_0}$ . Thus if  $\varphi$  is a capacity test function for  $\tilde{\Sigma}_{i_0}$ , i.e.  $\varphi(\Sigma_{i_0}) = 1$  and  $\varphi \rightarrow 0$  at infinity, then  $\varphi$  is also a capacity test function for  $\Sigma_i$ . Since  $C(\Sigma_i)$  is taking the infimum over a larger set of test functions than  $C(\tilde{\Sigma}_{i_0})$ ,  $C(\Sigma_i) \leq C(\tilde{\Sigma}_{i_0})$ . Thus if we create a new sequence of surfaces  $\bar{\Sigma}_i$ , alternately choosing from  $\Sigma_i$  and  $\tilde{\Sigma}_i$ , such that each surface contains the next we get a nonincreasing sequence of capacities. Thus if either original sequence of surfaces has a limit of capacity, then this new sequence must as well, and it must be the same. Hence,  $\lim_{i \rightarrow \infty} C(\Sigma_i) = \lim_{i \rightarrow \infty} C(\tilde{\Sigma}_i)$ .  $\square$

Now we look at the relationship between capacity and the Hawking mass of a surface. We will use techniques similar to those used in [4].

**Theorem 5.2.4.** *Let  $M$  be an asymptotically flat 3 manifold with nonnegative scalar curvature, and negative point mass singularity  $p$ . Let  $\Sigma_i$  be a family of surfaces converging to  $p$ . Assume each  $\Sigma_i$  is a minimizing hull. Assume the areas of  $\Sigma_i$  are going to zero. Then if the Hawking mass of the surfaces is bounded below, the capacities of surfaces of foliation near  $p$  must go to zero.*

*Proof.* To use Geroch monotonicity, we need to know that our IMCF surfaces stay connected. In the weak formulation of IMCF, the level sets  $\Sigma_t$  always bound a region in  $\bar{M}$ . Thus if  $\Sigma_t$  is not connected, one of its components  $\Sigma_t^*$  must not bound

a region. That is,  $\Sigma_t^*$  is not homotopic to a point in  $M$ . Since  $M$  is smooth, it must have finite topology on any bounded set. Thus we know that near  $p$ , there is a minimum size for a surface that does not bound a region. Call this size  $A_{\min}$ . Thus if we have any surface that does not bound a region, it must have area greater than  $A_{\min}$ . According to Lemma 4.3.1 the area of our surfaces grow exponentially. Thus if we restrict ourselves to starting IMCF with a surface with area  $A_{\min}/e$ , and only run the flow for time 1, we will stay connected. At first glance it seems we may need to worry about the jumps in weak IMCF, however Geroch monotonicity doesn't depend on smoothness of the flow, and neither does the area growth formula. Lemma 4.3.1 holds from  $t = 0$ . Thus even with jumps, the area of our surfaces will remain below  $A_{\min}$ .

Now recall that capacity of a surface is defined by

$$C(\Sigma) = \inf \left\{ \int_M \|\nabla \varphi\|^2 dV \mid \varphi(\Sigma) = 1, \varphi(\infty) = 0 \right\}. \quad (5.33)$$

Here, the integral is only over the portion of  $M$  outside of  $\Sigma$ . Call this integral,  $\mathcal{E}(\varphi)$ , the *energy* of  $\varphi$ . Thus for any  $\varphi$  with  $\varphi(\infty) = 0$  and  $\varphi(\Sigma) = 1$  we have  $\mathcal{E}(\varphi) \geq C(\Sigma)$ . So we will find an estimate that relates the Hawking mass and the energy of a test function  $\varphi$ .

Choose a starting surface  $\Sigma$  with sufficiently small starting area. Let  $f$  be the level set function of the associated weak IMCF starting with the surface  $\Sigma$ . Call the resulting level sets  $\Sigma_t$ . Now if we use a test function of the form  $\varphi = u(f)$ , then the energy of  $\varphi$  is given by

$$\mathcal{E}(\varphi) = \int_M \|\nabla f\|^2 (u')^2 dV. \quad (5.34)$$

Since  $f$  is given by IMCF, we know that  $\|\nabla f\| = H$  where  $H$  is the mean curvature of the level sets. Next we will use the co-area formula. This states that if we have a function  $z$  on a domain  $\Omega$ , and a function  $h : \mathbb{R} \rightarrow \mathbb{R}$  so that the range of  $h(z)$  is  $[a, b]$ , then

$$\int_{\Omega} h dV = \int_a^b h(z(t)) \int_{S_t} |\nabla z(t)| dA_t dt. \quad (5.35)$$

Here  $S_t$  are the level sets of  $h(z(t))$ . If we use the co-area formula with the foliation  $\Sigma_t$ , our integral becomes

$$\mathcal{E}(\varphi) = \int_0^\infty (u'(t))^2 \int_{\Sigma_t} |H| dA_t dt. \quad (5.36)$$

Here the co-area gradient term cancels one of the  $\|\nabla f\|$  terms. Now we will bound the interior integral of curvature. We know that IMCF causes the Hawking mass to be nondecreasing in  $t$ . We first rewrite the definition of the Hawking mass  $m_H(\Sigma_t^i) = m(t)$  as:

$$\int H^2 dA_t = 16\pi \left( 1 - m(t) \sqrt{\frac{16\pi}{A(t)}} \right). \quad (5.37)$$

Here  $A(t)$  is the area of  $\Sigma_t$ . Since the Hawking mass is nondecreasing under IMCF we have:

$$\int H^2 dA_t \leq 16\pi \left( 1 - m(0) \sqrt{\frac{16\pi}{A(t)}} \right). \quad (5.38)$$

Thus we can use Cauchy-Schwartz to get:

$$\int |H| dA_t \leq \sqrt{A(t)} \sqrt{16\pi \left( 1 - m(0) \sqrt{\frac{16\pi}{A(t)}} \right)}. \quad (5.39)$$

We can rewrite this as:

$$\int |H| dA_t \leq \sqrt{\alpha A(t) + \beta \sqrt{A(t)}}. \quad (5.40)$$

Furthermore, since  $A(t)$  grows exponentially in  $t$ , we can write this as:

$$\int |H| dA_t \leq \sqrt{\alpha e^t + \beta e^{t/2}} = v(t). \quad (5.41)$$

Where  $A_0$  has been absorbed into  $\alpha$  and  $\beta$ . Thus our energy formula has become

$$\mathcal{E}(\varphi) \leq \int_0^\infty (u'(t))^2 v(t) dt. \quad (5.42)$$

with

$$v(t) = \sqrt{\alpha e^t + \beta e^{t/2}} \quad (5.43)$$

where  $\alpha = 16\pi A_0$ ,  $\beta = (16\pi)^{3/2} A_0^{1/2} |m_0|$ , and  $A_0$  is  $A(\Sigma_0)$ . This means we can pick

our test function  $u(t)$  to be as simple as:

$$u(t) = \begin{cases} 1-t & 0 \leq t \leq 1 \\ 0 & t \geq 1 \end{cases} \quad (5.44)$$

Then our integral becomes:

$$\begin{aligned} E(\varphi) &\leq \int_0^1 v(t) dt \\ &= \int_0^1 \sqrt{\alpha e^t + \beta e^{t/2}} dt \\ &= \int_0^1 e^{t/4} \sqrt{\alpha e^{t/2} + \beta} dt \\ &= 4 \int_1^{e^{1/4}} \sqrt{\alpha x^2 + \beta} dx \\ &\leq 4 \int_1^{e^{1/4}} \sqrt{\alpha} x + \sqrt{\beta} \\ &= 2\sqrt{\alpha}(e^{1/2} - 1) + 4\sqrt{\beta}(e^{1/4} - 1) \\ &\leq 2\sqrt{\alpha} + 2\sqrt{\beta}. \end{aligned}$$

Since  $m_H(\Sigma) \leq \sqrt{\frac{|\Sigma|}{16\pi}}$ ,  $m_0$  is bounded above. By assumption  $m_0$  is bounded below, so  $\alpha$  and  $\beta$  are bounded by multiples of  $A_0$  and  $\sqrt{A_0}$  respectively. Thus  $\mathcal{E}(\varphi)$  goes to zero if  $A_0 \rightarrow 0$  and  $m_0$  is bounded. Hence  $C(p)$  must be zero since it is the infimum over a positive set with elements approaching zero.  $\square$

**Theorem 5.2.5** (Capacity Theorem). *Let  $M$  be an asymptotically flat 3 manifold with nonnegative scalar curvature, and negative point mass singularity  $p$ , such that there exists a family of surfaces,  $\Sigma_i$ , converging to  $p$  with area going to zero. Then if the capacity of  $p$  is nonzero, the Hawking masses of the surfaces  $\Sigma_i$  must go to  $-\infty$ .*

*Proof.* Any such family of surfaces will generate a family  $\Sigma'_i$  of minimizing hulls that will also converge to  $p$ . By Lemma 5.2.4, the masses of  $\{\Sigma'_i\}$  must go to  $-\infty$ . Thus the masses of  $\{\Sigma'_i\}$  must go to  $-\infty$ . Thus for sufficiently large  $i$ , the masses of the minimizing hulls are all negative. From then on Lemma 5.1.3 applies, and the masses of  $\Sigma_i$  must be less than the masses of  $\Sigma'_i$ . Hence they also converge to  $-\infty$ .  $\square$

# Chapter 6

## Symmetric Singularities

In this chapter we will look at some more examples of negative point mass singularities. The symmetry ansatz provides more structure than in the definition of a NPMS.

### 6.1 Spherical Solutions

A spherically symmetric manifold,  $M$ , has a metric given by

$$ds^2 = dr^2 + \frac{A(r)}{4\pi} dS^2. \quad (6.1)$$

We can directly calculate that the scalar curvature of  $M$  is given by

$$R = \frac{16\pi A + A'^2 - 4AA''}{2A^2}. \quad (6.2)$$

For this manifold to be asymptotically flat it is necessary for the Hawking masses of the coordinate spheres to approach a constant. The Hawking mass of a coordinate sphere is given by

$$m_H(S) = \sqrt{\frac{A}{16\pi}} \left( 1 - \frac{1}{16\pi} \frac{A'^2}{A} \right). \quad (6.3)$$

Due to the spherical symmetry of the manifold, we know that if IMCF is started with coordinate spheres, it must continue with coordinate spheres. Thus we know that this quantity must be non-decreasing. For this manifold to be asymptotically flat, this quantity must have a limit at  $\infty$ . This limit is the ADM mass.

The only possible location for a singularity in such a manifold is at the origin. We can find a straight forward function to resolve the singularity. We need a smooth function  $\varphi$  such that

$$\lim_{r \rightarrow \infty} \frac{A(r)}{\varphi^4} = 4\pi, \quad (6.4)$$

or any other constant.

Multiplying by  $\varphi^{-4}$  to find the model space changes our metric to the form

$$\tilde{d}s^2 = d\rho^2 + \frac{\tilde{A}(\rho)}{4\pi} dS^2. \quad (6.5)$$

Where  $\tilde{A}$  goes to  $4\pi$  as  $\rho$  approaches zero. Let  $\Sigma$  be the surface  $\rho = 0$ . The behavior of  $\varphi$  away from the singularity is very flexible as long it is bounded, nonzero, and goes to one at infinity. In order for this to be a regular singularity, we need that  $\rho$  be well defined. Thus we must require

$$\rho(r) = \int_0^r \varphi^{-2} dr = \int_0^r \frac{dr}{A^{1/2}(r)} < \infty \quad (6.6)$$

for finite  $r$ . Any spherically symmetric singularity with this condition on  $A(r)$  must be regular. The regular mass is given by

$$m_R = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma} \tilde{\nu}(\varphi)^{4/3} d\tilde{A} \right)^{3/2}. \quad (6.7)$$

Since we have only defined the asymptotic behavior of  $\varphi$  near  $r = 0$ , we will compute everything as limits as  $r$  goes to zero. First we need to find  $\tilde{\nu}(\varphi)$ . Since  $\rho = \int_0^r \varphi^{-2} dr$ ,

a chain rule calculation of  $\frac{\partial \varphi}{\partial \rho}$  yields

$$\tilde{\nu}(\varphi) = \varphi^2 \frac{\partial \varphi}{\partial r}. \quad (6.8)$$

Putting this in and expanding the other terms in the definition of the regular mass gives us

$$m_R = -\lim_{r \rightarrow 0} \frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma_r} \left( \varphi^2 \frac{\partial \varphi}{\partial r} \right)^{4/3} \varphi^{-4} dA \right)^{3/2} \quad (6.9)$$

$$= -\lim_{r \rightarrow 0} \frac{1}{4} \frac{1}{\pi^{3/2}} A^{3/2}(r) \varphi^{-2} \left( \frac{\partial \varphi}{\partial r} \right)^2 \quad (6.10)$$

Using equation (6.4) and l'Hôpital's rule we find that

$$\frac{\partial \varphi}{\partial r} = \frac{1}{16\pi\varphi^3} \frac{\partial A}{\partial r} \quad (6.11)$$

Continuing from above

$$\begin{aligned} m_R &= -\lim_{r \rightarrow 0} \frac{1}{4} \frac{1}{\pi^{3/2}} \frac{1}{256\pi^2} \varphi^{-8} A^{3/2} A'^2 \\ &= -\lim_{r \rightarrow 0} \frac{1}{1024\pi^{7/2}} \frac{16\pi^2}{A^2} A^{3/2} A'^2 \\ &= -\lim_{r \rightarrow 0} \frac{1}{64\pi^{3/2}} \frac{A'^2}{A^{1/2}}. \end{aligned} \quad (6.12)$$

This agrees with the limit of the Hawking masses of coordinate spheres as  $r \rightarrow 0$  and hence  $A \rightarrow 0$  as well.

For completeness, we can also examine the capacity of the central point in these solutions. The capacity of a coordinate sphere in an asymptotically flat spherically symmetric manifold is given by the  $1/\rho$  term in of the harmonic function that is 1 on the sphere and 0 at infinity. Since harmonic functions have constant flux with

respect to the outward normal, we could also describe this function as the constant flux function which goes to 0 at infinity and 1 on the coordinate sphere. Then the capacity of the sphere is given by the flux constant of this function, divided by  $-4\pi$ . Reversing this definition we define the following function:

**Definition 6.1.1.** Let  $(M, g)$  be an asymptotically flat spherically symmetric manifold. Let  $f$  be the radial function that has constant outward flux  $-4\pi$  through coordinate spheres and goes to zero at infinity. Call  $f$  the *radial capacity function* for  $(M, g)$ .

This definition allows the following lemma.

**Lemma 6.1.2.** *Let  $f(r)$  be the radial capacity function for the manifold  $(M, g)$ . Then the capacity of the coordinate sphere at  $r = r_0$  is given by  $f(r_0)^{-1}$ .*

*Proof.* Consider the function  $f(r)/f(r_0)$ . This function is 1 on the coordinate sphere  $r = r_0$ , goes to zero at infinity, and is harmonic. In the asymptotic regime its  $1/\rho$  coefficient is  $1/f(r_0)$ . Thus this is the capacity of the sphere  $r = r_0$ .  $\square$

Thus the capacity of the central point is given by  $\lim_{r \rightarrow 0} f(r)^{-1}$ . To calculate this value we first note that

$$\frac{df}{dr} = -\frac{4\pi}{A(r)} \quad (6.13)$$

since  $f$  has constant flux  $-4\pi$ . Thus

$$f(r) = -\int_r^\infty f'(r)dr = 4\pi \int_r^\infty \frac{dr}{A(r)}. \quad (6.14)$$

In order for the capacity of the central point to be nonzero, this must be finite. Asymptotic flatness tells us that the part of the integral in the asymptotic regime is

finite. Thus the only concern is where  $r \rightarrow 0$  and hence  $A(r) \rightarrow 0$ . Thus if

$$\int_0^\epsilon \frac{dr}{A(r)} \quad (6.15)$$

is finite, our central point has positive capacity. For example, if we assume that as  $r \rightarrow 0$ ,  $A(r)$  is asymptotically a multiple of a power of  $r$ , as  $kr^p$ . Then the capacity is positive exactly when  $p < 1$ . In this case, we see by equation (6.12) that the mass of the singularity is infinite, confirming Theorem 5.2.5. However if  $1 \leq p < 4/3$ , we see that the mass of our singularity is still infinite, but the capacity is now finite. This removes the possibility of strengthening Theorem 5.2.5 into an if and only if without additional hypotheses.

## 6.2 Overview of Weyl Solution

We will be looking at axisymmetric static vacuum spacetimes. These examples have two Killing fields, one spacelike and one timelike. These reflect the rotational and time translation symmetry of our spacetime. Furthermore the timelike Killing field is hypersurface orthogonal, and the two Killing fields commute. Since our two vector fields commute, we can call the timelike field  $\partial_t$  and the spacelike field  $\partial_\theta$ . Since they are Killing fields, we know the metric is only a function of the remaining two coordinates  $x_1, x_2$ . We can also assume that  $\partial_\theta$  is orthogonal to  $\partial_t$ . Thus our metric is of the form

$$g = -A^2 dt^2 + B^2 d\theta^2 + g_{11} dx_1^2 + g_{12} dx_1 dx_2 + g_{22} dx_2^2. \quad (6.16)$$

Here  $A, B, g_{ij}$  are functions of  $x_1$  and  $x_2$ . We set  $x_1 = AB$ , and chose  $x_2$  to be orthogonal to the other coordinates. Renaming  $x_1, \rho$  and  $x_2, z$ , our metric takes the

form

$$g = -A^2 dt^2 + \rho^2 A^{-2} d\theta^2 + U^2 d\rho^2 + V^2 dz^2. \quad (6.17)$$

See Theorem 7.1.1 in [12] for an explanation of the lack of cross terms and further details on this derivation.

We may also rescale  $z$  to set  $V = U$ . Our metric is now encoded in the functions  $A(\rho, z)$  and  $U(\rho, z)$ . If we define  $\lambda$  and  $\mu$  by

$$\lambda = \ln A \quad \mu = \ln(AU) \quad (6.18)$$

our metric looks like

$$g = -e^{2\lambda} dt^2 + e^{-2\lambda} [\rho^2 d\theta^2 + e^{2\mu} (d\rho^2 + dz^2)]. \quad (6.19)$$

Now, if we compute the curvature of this metric, and apply the vacuum condition we get the following equations for  $\lambda$  and  $\mu$ :

$$0 = \lambda_{\rho\rho} + \frac{1}{\rho} \lambda_{\rho} + \lambda_{zz} \quad (6.20)$$

$$\mu_{\rho} = \rho (\lambda_{\rho}^2 - \lambda_z^2) \quad (6.21)$$

$$\mu_z = 2\rho \lambda_{\rho} \lambda_z. \quad (6.22)$$

The first one is the same as the statement that  $\Delta\lambda = 0$  when viewed as a function of flat  $\mathbb{R}^3$  with coordinates  $(r, z, \theta)$ . We can use this to generate spacetimes. We can think of  $\lambda$  as a classical potential function. However, we should not think that  $g$  gives the metric with this gravitational potential. For example, the Schwarzschild solution is generated by a  $\lambda$  that is not spherically symmetric

As always we are interested in asymptotically flat spacetimes. In this case we want our metric to be the flat metric in cylindrical coordinates at infinity. Thus we

need the following asymptotics on  $\lambda$  and  $\mu$ :

$$\lim_{r \rightarrow \infty} \lambda = 0 \quad (6.23)$$

$$\lim_{r \rightarrow \infty} \mu = 0 \quad (6.24)$$

Where  $r^2 = \rho^2 + z^2$ . Since  $\lambda$  is flat-harmonic, we know that at infinity it looks like  $C|r|^{-1} + \mathcal{O}(r^{-2})$ . This decay and equation (6.20) tells us that

$$\lim_{r \rightarrow \infty} |\lambda_\rho|, |\lambda_z| \leq \frac{C}{r^2}. \quad (6.25)$$

Putting those into equations (6.21) and (6.22) gives us

$$\lim_{r \rightarrow \infty} |\mu_\rho|, |\mu_z| \leq \frac{C}{r^3}. \quad (6.26)$$

These conditions are enough for asymptotic flatness. Thus all that is required of our metric for it to be asymptotically flat is that  $\lambda$  and  $\mu$  approach zero at  $\infty$ . To calculate the ADM mass of  $g$ , we need coordinates that are asymptotically Cartesian rather than asymptotically cylindrical. Using the change of coordinates

$$x = \rho \cos \theta \quad y = \rho \sin \theta \quad (6.27)$$

we get that our metric is

$$\begin{aligned} g = & -e^{2\lambda} dt^2 + e^{-2\lambda} \left( 1 + \frac{(e^{2\mu} - 1)x^2}{x^2 + y^2} \right) dx^2 + \frac{e^{-2\lambda}(e^{2\mu} - 1)xy}{x^2 + y^2} dx dy \\ & + e^{-2\lambda} \left( 1 + \frac{(e^{2\mu} - 1)y^2}{x^2 + y^2} \right) dy^2 + e^{-2\lambda} e^{2\mu} dz^2 \end{aligned} \quad (6.28)$$

If we plug this into the formula for the ADM mass we get the following:

$$m = \lim_{r \rightarrow \infty} \int_{S_r} \frac{e^{-2\lambda}}{16\pi r} [1 - 2z\lambda_z - 2\rho\lambda_\rho + e^{2\mu}(2\rho\mu_\rho + 2z\mu_z - 2\rho\lambda_\rho - 2z\lambda_z - 1)] dA_\delta \quad (6.29)$$

As  $r$  grows the terms  $\lambda_z$  and  $\lambda_\rho$  are at most order  $r^{-2}$ . The derivatives of  $\mu$  are at most order  $r^{-3}$ . The function  $e^{-2\lambda}$  is 1, as is the function  $e^{2\mu}$ . Therefore this integral is

$$m = \frac{-1}{4\pi} \lim_{r \rightarrow \infty} \int_{S_r} \frac{1}{r} (z\lambda_z + \rho\lambda_\rho) dA_\delta = \frac{-1}{4\pi} \lim_{r \rightarrow \infty} \int_{S_r} \langle \nu, \nabla \lambda \rangle_\delta dA_\delta. \quad (6.30)$$

Since  $\lambda$  is harmonic in the flat metric, we can compute this on any surface homotopic to a large sphere at infinity.

**Lemma 6.2.1.** *Let  $(M, g)$  be the  $t = 0$  slice of an asymptotically flat axisymmetric vacuum static manifold with metric*

$$g = e^{-2\lambda} [\rho^2 d\theta^2 + e^{2\mu} (d\rho^2 + dz^2)]. \quad (6.31)$$

*Then the ADM mass of  $(M, g)$  is given by*

$$-\frac{1}{4\pi} \lim_{r \rightarrow \infty} \int_{\Sigma} \langle \nu, \nabla \lambda \rangle_\delta dA_\delta, \quad (6.32)$$

*where  $\Sigma$  is any surface enclosing the singularities of  $\lambda$ .*

As in the spherically symmetric case, a useful harmonic function can tell us about the capacity of the singular points. In this case our metric comes with a harmonic function. The function  $e^\lambda$  is harmonic in our metric. It goes to 1 at infinity and infinity at any positive singularities of our potential. Thus the capacity of level sets of  $e^\lambda$  must go to zero as  $\lambda \rightarrow \infty$ . This tells us that these singularities have zero capacity.

### 6.3 Zippoy–Voorhees Metrics

The particular family of metrics we will consider are the Zipoy–Voorhees, or  $\gamma$  metrics. These are given by the potential arising from a uniform density rod of length  $2a$  and mass  $m$  at the origin. This gives a potential of

$$\lambda = \frac{m}{2a} \ln \frac{R_+ + R_- - 2a}{R_+ + R_- + 2a} \quad \mu = -\frac{m^2}{2a^2} \ln \frac{4R_+R_-}{(R_+ + R_-)^2 - 4a^2}. \quad (6.33)$$

Where  $R_{\pm} = \sqrt{\rho^2 + (z \pm a)^2}$ . If  $m = 2a$ , then this metric is Schwarzschild. It only represents the area outside the horizon. The interval  $[-a, a]$  on the  $z$ -axis is the event horizon. If  $m \neq 2a$  then this metric has a naked singularity at  $\rho = 0$ ,  $|z| \leq a$ . See [8] for more information on the cases with positive  $m$ . When  $m \neq 0, 2a$ , the resulting spacetime has ADM mass  $m$  and the  $[-a, a]$  interval on the  $z$ -axis becomes a candidate for a negative mass singularity. To investigate the area near the singularity, we will look at  $\rho$  constant cylinders from  $z = -a$  to  $a$ . The area of these cylinders is given by

$$2\pi\rho \int_{-a}^a e^{\mu-2\lambda} dz. \quad (6.34)$$

Since both  $\lambda$  and  $\mu$  are given by logs, the integral simplifies to

$$2\pi\rho \int_{-a}^a \left( \frac{4R_+R_-}{(R_+ + R_-)^2 - 4a^2} \right)^{-m^2/2a^2} \left( \frac{R_+ + R_- - 2a}{R_+ + R_- + 2a} \right)^{-m/a} dz. \quad (6.35)$$

When  $z$  is between  $-a$  and  $a$ , and  $\rho$  is small we have the following approximations:

$$\frac{4R_+R_-}{(R_+ + R_-)^2 - 4a^2} = \frac{(a^2 - z^2)^2}{a^2} \frac{1}{\rho^2} + \mathcal{O}(1) \quad (6.36)$$

$$\frac{R_+ + R_- - 2a}{R_+ + R_- + 2a} = \frac{1}{4(a^2 - z^2)} \rho^2 + \mathcal{O}(\rho^4). \quad (6.37)$$

Hence our integral becomes

$$2\pi\rho^{m^2/a^2-2m/a+1} 4^{m/a} \int_{-a}^a (a^2 - z^2)^{-m^2/a^2+m/a} a^{m^2/a^2} dz. \quad (6.38)$$

Now if  $m \neq a$ , the  $\rho$  term has positive exponent. Thus the areas of these surfaces go to zero. The fact that the function  $e^\lambda$  is harmonic and goes to infinity near the  $[-a, a]$  on the  $z$  axis tells us that these surfaces have zero capacity. Thus, as long as  $m \neq a$ , these fulfill the definition of a negative point mass singularity.

Continuing in this fashion we can estimate the mass of this singularity. We will just estimate the mass of the singularities using the level sets of the function  $\lambda$ . Referring back to Definition 2.3.4, our function  $h_i$  given by the level set  $\lambda = L$  is given by

$$h_i = \frac{L}{L-1} - \frac{e^\lambda}{L-1}. \quad (6.39)$$

Here  $L = \pm i$ , with the sign chosen to be the opposite sign to  $m$ . We can calculate  $\nu(h_i)^{4/3}$  as

$$\nu(h_i)^{4/3} = \frac{1}{(L-1)^{4/3}} e^{\frac{4}{3}\lambda - \frac{4}{3}\mu} (\lambda_\rho^2 + \lambda_z^2)^{2/3}. \quad (6.40)$$

Now we will approximate the surface  $\lambda = L$  by a level set  $\rho = \rho_i$ . Noting that  $\lambda_\rho$  is much larger than  $\lambda_z$  tells us that this assumption is valid. With that assumption, our mass integral becomes

$$\mathcal{E} = \frac{2\pi\rho}{(e^\lambda - 1)^{4/3}} e^{-\frac{2}{3}\lambda} \int_{-a}^a e^{-\frac{1}{3}\mu} (\lambda_\rho^2 + \lambda_z^2)^{2/3} dz. \quad (6.41)$$

Now we note the first order behavior of  $\lambda$  and  $\mu$  near  $\rho = 0, |z| < a$ .

$$\begin{aligned} \lambda &\sim -\frac{m}{a} \ln \rho & \mu &\sim \frac{m^2}{a^2} \ln \rho \\ \lambda_\rho &\sim -\frac{m}{a} \frac{1}{\rho} & \lambda_z &\sim \frac{mz}{a(z^2 - a^2)}. \end{aligned} \quad (6.42)$$

Using these we approximate the above integral

$$\mathcal{E} \sim \frac{2\pi\rho}{(\rho^{-m/a} - 1)^{4/3}} \rho^{\frac{2}{3}\frac{m}{a}} \int_{-a}^a \rho^{-\frac{1}{3}\frac{m^2}{a^2}} \left( \frac{m^2}{a^2} \rho^{-2} + \frac{m^2 z^2}{a^2(z^2 - a^2)^2} \right)^{2/3} dz. \quad (6.43)$$

As  $\rho$  goes to zero, this has  $\rho$  dependence as

$$\mathcal{E} \sim C \cdot \frac{\rho^{\frac{2}{3}\frac{m^2}{a^2} - \frac{1}{3}\frac{m}{a} - 1}}{(\rho^{-m/a} - 1)^{4/3}}. \quad (6.44)$$

For some constant  $C$ . If  $m > 0$ , then the bottom term contributes a  $\rho^{\frac{4}{3}\frac{m}{a}}$  to the growth giving an overall power of  $\frac{2}{3}\frac{m^2}{a^2} + \frac{m}{a} - 1$ . Otherwise it contributes negligibly.

Thus we have the following  $\rho$  dependence

$$\mathcal{E} \sim \begin{cases} C_+ \rho^{\frac{2}{3}\frac{m^2}{a^2} + \frac{m}{a} - 1} & m > 0 \\ C_- \rho^{\frac{2}{3}\frac{m^2}{a^2} - \frac{1}{3}\frac{m}{a} - 1} & m < 0. \end{cases} \quad (6.45)$$

The exponent on the  $\rho$  is negative when  $\frac{m}{a} \in \left(-1, \frac{\sqrt{33}}{4} - \frac{3}{4}\right)$  and negative when  $\frac{m}{a}$  falls outside the closure of that range. Outside that range we have produced an example of a set of surfaces which give zero to the mass under Definition 2.3.4. Hence, since that mass is a sup over all such surfaces, we know it must be zero. For  $\frac{m}{a}$  inside that range, our surfaces give a mass of  $-\infty$ . This is inconclusive, since it is entirely possible that there exists a better behaved family of surfaces.

Checking our two known cases,  $\frac{m}{a} = \pm 1$  we see that the positive Schwarzschild metric doesn't have a singularity, and the mass of the negative Schwarzschild is nonzero and finite. It shouldn't be surprising that for positive  $m$  outside that range we do not get a negative mass as a Negative Point Mass Singularity since these singularities are the only points without zero scalar curvature in a static manifold with positive ADM mass. It seems sensible that they shouldn't be assigned a negative mass.

# Chapter 7

## Open Questions

There are a number of unanswered questions having to do with negative point mass singularities. The most prominent is extending Theorem 5.1.8 to include multiple singularities. Bray, in [6], has a solution that depends on an unproven geometric conjecture. A further result would be what I have been calling the “Mixed Penrose Inequality.” This would be a result that combines the two cases, singularities and horizons, and provides a lower bound on the ADM mass of a manifold containing horizons and singularities. The desired inequality is presented in Appendix B.

It is clear that a removable singularity should have mass zero. Precisely what conditions on an negative point mass singularity guarantee that it is removable requires further investigation.

On a more concrete note the Zipoy–Voorhees metrics with  $\frac{m}{a} \in \left(-1, \frac{\sqrt{33}}{4} - \frac{3}{4}\right)$  deserve further investigation. In particular the metrics with  $\frac{m}{a}$  in that range and positive seem to offer the possibility of negative mass singularities in a vacuum static manifold with positive ADM mass.

On a broader scale, the current presentation of negative point mass singularities is only in the Riemannian case. Extending the definitions to the full Lorentzian context would require some equation to control the behavior of the manifold in the neighborhood of the singularity in the timelike direction. Furthermore, a new definition of mass would have to be added, since the current definition depends on the foliation of surfaces converging to the singularity. Furthermore, the conformal factor definition of the regular case doesn't translate in an obvious way to the Lorentzian case even with the Schwarzschild metric. Studying negative point mass singularities in a spacetime is an important direction to pursue.

# Appendix A

## Miscellaneous Calculations

### A.1 Calculation of total magnification of a NPMS lens

The magnification of a negative point mass singularity lens is given by

$$\mu = \frac{1}{1 - \frac{m^2}{\|x\|^4}} = \frac{\|x\|^4}{\|x\|^4 - m^2} = 1 + \frac{m^2}{\|x\|^4 - m^2}. \quad (\text{A.1})$$

The image locations  $x_{\pm}$  associated to a given  $y$  are

$$x_{\pm} = \frac{1}{2} \left( y \pm \sqrt{y^2 + 4m} \right). \quad (\text{A.2})$$

First we calculate  $\|x_{\pm}\|^4$  as

$$\|x_{\pm}\|^4 = \frac{1}{16} \left( y \pm \sqrt{y^2 + 4m} \right)^4 \quad (\text{A.3})$$

$$= \frac{1}{16} \left( y^2 \pm 2y\sqrt{y^2 + 4m} + y^2 + 4m \right)^2 \quad (\text{A.4})$$

$$= \frac{1}{16} \left( 4y^4 + 16y^2m + 16m^2 + 4y^2(y^2 + 4m) \pm (2y^2 + 4m)4y\sqrt{y^2 + 4m} \right) \quad (\text{A.5})$$

$$= \frac{1}{16} \left( 8y^4 + 32y^2m + 16m^2 \pm (8y^3 + 16ym) \sqrt{y^2 + 4m} \right) \quad (\text{A.6})$$

$$= \frac{1}{2}y^4 + 2y^2m + m^2 \pm \left( \frac{1}{2}y^3 + ym \right) \sqrt{y^2 + 4m}. \quad (\text{A.7})$$

Now we define the following terms

$$A = \frac{1}{2}y^4 + 2y^2m \quad (\text{A.8})$$

$$B = m^2 \quad (\text{A.9})$$

$$C = y \left( \frac{1}{2}y^2 + m \right) \sqrt{y^2 + 4m}. \quad (\text{A.10})$$

Thus  $\|x_{\pm}\|^4 = A + B \pm C$ . The negative image has negative magnification, so we have to look at  $\mu(x_+) - \mu(x_-)$ . This is given by

$$\mu_t(x) = \frac{\|x_+\|^4}{\|x_+\|^4 - m^2} - \frac{\|x_-\|^4}{\|x_-\|^4 - m^2} \quad (\text{A.11})$$

$$= \frac{A + B + C}{A + C} - \frac{A + B - C}{A - C} \quad (\text{A.12})$$

$$= 1 + \frac{B}{A + C} - \left( 1 + \frac{B}{A - C} \right) \quad (\text{A.13})$$

$$= \frac{-2BC}{A^2 - C^2}. \quad (\text{A.14})$$

We can simplify this as

$$2BC = ym^2 (y^2 + 2m) \sqrt{y^2 + 4m} \quad (\text{A.15})$$

$$A^2 - C^2 = \left( \frac{1}{2}y^4 + 2y^2m \right)^2 - y^2 \left( \left( \frac{1}{2}y^2 + m \right) \sqrt{y^2 + 4m} \right)^2 \quad (\text{A.16})$$

$$= -y^4m^2 - 4y^2m^3 = -y^2m^2(y^2 + 4m). \quad (\text{A.17})$$

Thus we get

$$\mu_t = \frac{y^2 + 2m}{y\sqrt{y^2 + 4m}}. \quad (\text{A.18})$$

## A.2 Solutions to Cusp Equation

In the following we will drop the  $*$  subscripts on the quantities  $Z$ ,  $\gamma$ , and  $m$ . We are solving the equation

$$0 = Z + \epsilon_\kappa \left( \gamma + \frac{m}{\bar{z}^2} \right) \bar{Z}, \quad (\text{A.19})$$

where

$$Z = -4i \left( \gamma + \frac{m}{z^2} \right) \frac{m}{\bar{z}^3}, \quad (\text{A.20})$$

and

$$z = \pm \sqrt{\frac{m}{e^{-i\varphi} - \gamma}}. \quad (\text{A.21})$$

Thus we get the following

$$\begin{aligned} \gamma + \frac{m}{z^2} &= e^{-i\varphi} & \gamma + \frac{m}{\bar{z}^2} &= e^{i\varphi} \\ \frac{m^{3/2}}{z^3} &= (e^{-i\varphi} - \gamma)^{3/2} & \frac{\overline{m^{3/2}}}{\bar{z}^3} &= -\frac{m^{3/2}}{\bar{z}^3} = -(e^{i\varphi} - \gamma)^{3/2}. \end{aligned} \quad (\text{A.22})$$

Note the conjugate bar on the  $m$  on the last relation. Since  $m$  is negative  $m^{3/2}$  is imaginary, and so we had to multiply by  $-1$  when we conjugated the previous equation. Thus our equation is

$$0 = -4i \left( \gamma + \frac{m}{z^2} \right) \frac{m}{\bar{z}^3} + \epsilon_\kappa \left( \gamma + \frac{m}{\bar{z}^2} \right) 4i \left( \gamma + \frac{m}{z^2} \right) \frac{m}{z^3}. \quad (\text{A.23})$$

Multiplying by  $\frac{\sqrt{m}}{-4i}$  and using the substitutions in (A.22) and then multiplying by  $e^{-i\varphi/2}$  gives us

$$0 = e^{-\frac{3}{2}i\varphi} (e^{i\varphi} - \gamma)^{3/2} + \epsilon_\kappa e^{\frac{3}{2}i\varphi} (e^{-i\varphi} - \gamma)^{3/2}. \quad (\text{A.24})$$

If we define  $w = e^{-i\varphi} (e^{i\varphi} - \gamma)$ . Then we have

$$0 = w^{3/2} + \epsilon_\kappa \bar{w}^{3/2}. \quad (\text{A.25})$$

Which is solved when  $w^{3/2}$  purely real or imaginary, when  $\epsilon_\kappa$  is negative or positive respectively. This corresponds to  $w^3$  being purely real and having the opposite sign as  $\epsilon_\kappa$ . The imaginary part of  $w^3$  is

$$\text{Im}(w^3) = \gamma \sin \varphi [4\gamma^2 \cos^2 \varphi - 6\gamma \cos \varphi + 4 - \gamma^2]. \quad (\text{A.26})$$

Setting aside the case when  $\gamma = 0$ , we are left with the roots

$$\varphi_1 = 0, \quad \varphi_2 = \pi, \quad \varphi_{3,4} = \arccos \left( \frac{3 \pm \sqrt{4\gamma_*^2 - 3}}{4\gamma_*} \right), \quad \varphi_5 = 2\pi - \varphi_3, \quad \varphi_6 = 2\pi - \varphi_4. \quad (\text{A.27})$$

If  $\gamma^2 < 3/4$  we only have at most  $\gamma_1$  and  $\gamma_2$ . Looking at the real part of  $w^3$  now

$$\text{Re}(w^3) = -4\gamma^3 \cos^3 \varphi + 6\gamma^2 \cos^2 \varphi + (3\gamma^3 - 3\gamma) \cos \varphi - 3\gamma^2 + 1. \quad (\text{A.28})$$

Thus since there are no  $\sin \varphi$  terms, the real part of  $w^3$  only depends on  $\cos \varphi$ . For each  $\varphi_i$ , we only have a solution, and hence a cusp, for one value of  $\epsilon_\kappa$ , where  $\epsilon_\kappa$  has the same sign as  $w^3$ . Thus we need to find where the sign of  $w^3$  changes. All this is in Table A.1.

$\varphi_i$	$\cos(\varphi_i)$	$\text{Re}(w^3)$	+	-
$\varphi_1$	1	$-\gamma^3 + 3\gamma^2 - 3\gamma + 1 = (1 - \gamma)^3$	$\gamma < 1$	$\gamma > 1$
$\varphi_2$	-1	$\gamma^3 + 3\gamma^2 + 3\gamma + 1 = (1 + \gamma)^3$	always	
$\varphi_{3,5}$	$\frac{3 + \sqrt{4\gamma^2 - 3}}{4\gamma}$	$1 - \frac{3}{2}\gamma^2 + \frac{1}{2}\gamma^2\sqrt{4\gamma^2 - 3}$	$\gamma > 1$	$\sqrt{3/4} \leq \gamma < 1$
$\varphi_{4,6}$	$\frac{3 + \sqrt{4\gamma^2 + 3}}{4\gamma}$	$1 - \frac{3}{2}\gamma^2 - \frac{1}{2}\gamma^2\sqrt{4\gamma^2 - 3}$		always

**Table A.1:** Values of the real part of  $w^3$  for various  $\varphi_i$ .

# Appendix B

## Work Towards the Mixed Penrose Inequality

### B.1 Purpose

This appendix is a record of my attempts towards solving what we have been calling the “Mixed Penrose Inequality.” The desired conjecture is as follows:

**Conjecture B.1.1** (Mixed Penrose Inequality). *Let  $(M, g)$  be an asymptotically flat manifold with outer minimizing boundary  $\Sigma$  and negative mass singularities  $\bar{p}_i$ . Also assume that  $(M, g)$  has nonnegative scalar curvature. Then*

$$m_{ADM}(G) \geq \sqrt{\frac{|\Sigma|}{16\pi}} - \left( \sum_i m_R(p_i)^{2/3} \right)^{3/2}. \quad (\text{B.1})$$

The exponent on the negative point mass singularity term comes from consoli-

dating the integrals for the masses as follows (assuming they are regular):

$$-\left(\sum_i m_R[p_i]^{2/3}\right)^{3/2} = -\left(\sum_i \left[\frac{1}{4} \left(\frac{1}{\pi} \int_{\Sigma_i} \nu(\varphi)^{4/3} da\right)^{3/2}\right]^{2/3}\right)^{3/2} \quad (\text{B.2})$$

$$= -\left(\sum_i \frac{1}{4^{3/2}} \frac{1}{\pi} \int_{\Sigma_i} \nu(\varphi)^{4/3} da\right)^{3/2} \quad (\text{B.3})$$

$$= -\frac{1}{4} \left(\frac{1}{\pi} \int_{\cup_i \Sigma_i} \nu(\varphi)^{4/3} da\right)^{3/2}. \quad (\text{B.4})$$

This has so far been out of our reach. As a first case we have been working on a reduced conjecture which assumes the singularity and black hole have equal and opposite masses. Then all that is expected is that the ADM mass is positive.

**Conjecture B.1.2.** *Let  $(\overline{M}, \overline{g})$  be an asymptotically flat 3-manifold, with minimal boundary  $\overline{\Sigma}$  and negative mass singularity  $\overline{p}$  of opposite mass. Also assume that  $(\overline{M}, \overline{g})$  has nonnegative scalar curvature. Assume that  $\overline{p}$  can be resolved by a harmonic conformal factor  $\varphi$  so that  $(M, g)$  is asymptotically flat,  $\Sigma$  is still minimal, and  $\Pi$ , the resolution of  $p$ , is minimal as well. We can assume that  $\varphi = 1$  at infinity,  $\varphi = 0$  only on  $\Pi$ ,  $\nu(\varphi) = 1$  on  $\Pi$ , and  $\nu(\varphi) = 0$  on  $\Sigma$ .*

*Then the ADM mass of  $(\overline{M}, \overline{g})$  is non-negative.*

There are three methods we have for proving Penrose style theorems. Inverse mean curvature flow, as in [7], which only takes into account a single horizon or singularity, so probably won't be helpful here. Bray's conformal flow of metrics as in [5] is the method we worked most with. Lastly, Bray's Minimal Surface Resolution technique used in [6], might also be useful, however to use it, we would need to adapt

it to accommodate the black holes. Furthermore, one of the steps in its proof is not yet complete.

## B.2 Effects of Harmonic Conformal Flow

The basic idea here is to use a harmonic conformal flow on the model space  $(M, g)$ , which is equivalent to a harmonic conformal flow on the actual space  $(\overline{M}, \overline{g})$ .

If we look at a flow on  $\varphi$ , and set  $\psi = \frac{d}{dt}\varphi$ , then we can look at the changes to the various quantities. If we set  $m$  to be the ADM mass,  $M$  to be the mass of the black hole, and  $N$  to be the mass of the negative mass singularity.

$$N(\Pi) = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Pi} \nu(\varphi)^{4/3} dA \right)^{3/2} \quad (\text{B.5})$$

$$M(\Sigma) = \sqrt{\frac{|\overline{\Sigma}|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\overline{\Sigma}} H^2 d\overline{A} \right) \quad (\text{B.6})$$

For a first estimate we will look at the rate of change at time zero for initial  $\psi$ . First we look at  $\Pi$ . A direct calculation shows

$$\dot{N}(\Pi) = -\frac{3}{8} \left( \frac{1}{\pi} \int_{\Pi} \nu(\varphi)^{4/3} dA \right)^{1/2} \left( \frac{1}{\pi} \int_{\Pi} \nu(\varphi)^{1/3} \left[ \frac{4}{3} \nu(\psi) + H\psi \right] dA \right). \quad (\text{B.7})$$

Under the assumption that we are working at  $t = 0$ , we can put in the known information about  $\varphi$  to get:

$$\dot{N}(\Pi) = -\frac{3}{2} \left( \frac{1}{\pi} \int_{\Pi} \nu(\varphi)^{4/3} dA \right)^{1/2} \left( \frac{1}{\pi} \int_{\Pi} \nu(\varphi)^{1/3} \left[ \frac{4}{3} \nu(\psi) + H\psi \right] dA \right) \quad (\text{B.8})$$

$$= N(\Pi) \frac{\int_{\Pi} 2\nu(\psi) + \frac{3}{2} H\psi dA}{\int_{\Pi} dA}. \quad (\text{B.9})$$

If  $\Pi$  is minimal as well, we can drop the  $H$  term. We can also drop this term if we assume that  $\psi = 0$  on  $\Pi$ . Likewise we can compute the change to  $M$  as

$$\dot{M}(\Sigma) = \sqrt{\frac{1}{4\pi}} \left( \int_{\Sigma} \varphi^4 dA \right)^{-1/2} \left( \int_{\Sigma} \psi \varphi^3 dA \right) \quad (\text{B.10})$$

$$= M(\Sigma) \frac{\int_{\Sigma} 2\psi \varphi^3 dA}{\int_{\Sigma} \varphi^4 dA}. \quad (\text{B.11})$$

The integral of  $H^2$  term drops out since  $H$  is zero. We are also concerned with the change in the ADM mass of the manifold. We can measure that with the integral of the flux across a sphere at infinity, since this will measure the  $1/r$  term of  $\psi$ . (Here we assume that  $\psi = 0$  at infinity.) Since the harmonic functions have constant flux, we know that the flux across  $\Sigma$  and  $\Pi$  will also give us the  $1/r$  term for  $\psi$ . Thus the change in  $m$  is given by

$$\dot{m} = \frac{1}{4\pi} \left( \int_{\Sigma} \nabla(\psi) \cdot \nu dA + \int_{\Pi} \nabla(\psi) \cdot \nu dA \right) \quad (\text{B.12})$$

$$= \frac{1}{4\pi} \left( \int_{\Sigma} \nu(\psi) dA + \int_{\Pi} \nu(\psi) dA \right) \quad (\text{B.13})$$

Adding these together, the quantity we want to decrease is the *mass surplus*  $X$ ,

$$X = m - M - N \quad (\text{B.14})$$

$$\dot{X} = \dot{m} - \dot{M} - \dot{N} \quad (\text{B.15})$$

$$= \frac{1}{4\pi} \left( \int_{\Sigma} \nu(\psi) dA + \int_{\Pi} \nu(\psi) dA \right) - \quad (\text{B.16})$$

$$M \left( \frac{\int_{\Sigma} 2\psi \varphi^3 dA}{\int_{\Sigma} \varphi^4 dA} \right) - N \left( \frac{\int_{\Pi} 2\nu(\psi) + \frac{3}{2} H \psi dA}{\int_{\Pi} dA} \right). \quad (\text{B.17})$$

If we can flow so that this decreases until  $N$  (or  $M$ ) is zero, then we can apply the Penrose (NPMS) inequality to get that  $X \geq 0$  at this time, hence  $X \geq 0$  at  $t = 0$ ,

and we are done. At this point we tried different boundary conditions to accomplish this.

### B.2.1 Boundary Conditions 1

For a first try, set  $\psi$  by the boundary conditions  $\psi = 0$  at infinity,  $\psi = -\frac{\alpha}{2} \frac{f\varphi^4}{f\varphi^3}$  on  $\Sigma$ , with both integrals over  $\Sigma$ , and  $\psi = 0$  and  $\nu(\psi) = -\frac{\beta}{2}$  on  $\Pi$ . This gives a value for  $\dot{X}$  of

$$\dot{X} = \frac{1}{4\pi} \int_{\Sigma} \nu(\psi) dA - \frac{\beta}{8\pi} |\Pi| - \beta M - \alpha N. \quad (\text{B.18})$$

### B.2.2 Boundary Conditions 2

As a second try, we can look at the boundary conditions,  $\psi = 0$  at infinity,  $\psi = \alpha$  on  $\Sigma$ ,  $\nu(\psi) = \beta$  on  $\Pi$ , where we choose  $\alpha$  to be  $-\frac{f\varphi^4}{2f\varphi^3}$ , and pick  $\beta$  so that  $\dot{m}$  is zero. This means that the flux of  $\psi$  through the two surfaces is zero.

### B.2.3 Unknown Boundary Conditions

Trying the boundary conditions  $\psi = 0$  on  $\Pi$  and at infinity while we leave the boundary conditions of  $\psi$  on  $\Sigma$  up in the air for now, except that we want  $\psi < 0$  and  $\nu(\psi) > 0$ . If we set  $S$  to be a large sphere at infinity, then  $\dot{m}$  will be given by  $\frac{1}{4\pi} \int_S \nu(\psi)$ , with  $\nu$  being the outward pointing normal for all surfaces. Then since  $\psi$  is harmonic we know that

$$\int_S \nu(\psi) + \int_{\Sigma} \nu(\psi) + \int_{\Pi} \nu(\psi) = 0 \quad (\text{B.19})$$

Now we will calculate what we get for  $\dot{N}$  and for  $\int_{\Pi} \nu(\psi)$  for different values of  $\nu(\psi)$ .

Set  $\nu(\varphi) = c$ . Solving the formula for  $N$  for  $|\Pi|$  we get

$$N = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Pi} \nu(\varphi)^{4/3} dA \right)^{3/2} \quad (\text{B.20})$$

$$= -\frac{1}{4} \frac{1}{\pi^{3/2}} |\Pi|^{3/2} \nu(\varphi)^2 \quad (\text{B.21})$$

$$|\Pi| = \left( -4\pi^{3/2} N / c^2 \right)^{2/3} \quad (\text{B.22})$$

$$= 2^{4/3} \pi N^{2/3} c^{-4/3}. \quad (\text{B.23})$$

With this in hand we can compute  $\dot{N}$ .

$$\dot{N} = -\frac{3}{8} \left( \frac{1}{\pi} \int_{\Pi} \nu(\varphi)^{4/3} dA \right)^{1/2} \left( \frac{1}{\pi} \int_{\Pi} \frac{4}{3} \nu(\varphi)^{1/3} \nu(\psi) dA \right) \quad (\text{B.24})$$

$$= -\frac{|\Pi|^{1/2}}{2\pi^{3/2}} c \left( \int_{\Pi} \nu(\psi) dA \right) \quad (\text{B.25})$$

$$= -\frac{1}{2} \left( 2^{2/3} \pi^{1/2} N^{1/3} c^{-2/3} \right) \frac{1}{\pi^{3/2}} c \left( \int_{\Pi} \nu(\psi) dA \right) \quad (\text{B.26})$$

$$= -2^{-1/3} \pi^{-1} N^{1/3} c^{1/3} \left( \int_{\Pi} \nu(\psi) dA \right). \quad (\text{B.27})$$

If we choose  $c = -\frac{1}{32N}$ . Then

$$\dot{N} = -2^{-1/3} \pi^{-1} N^{1/3} c^{1/3} \left( \int_{\Pi} \nu(\psi) dA \right) \quad (\text{B.28})$$

$$= 2^{-1/3} \pi^{-1} N^{1/3} \left( \frac{1}{32N} \right)^{1/3} \left( \int_{\Pi} \nu(\psi) dA \right) \quad (\text{B.29})$$

$$= \frac{1}{4\pi} \left( \int_{\Pi} \nu(\psi) dA \right). \quad (\text{B.30})$$

Thus  $\dot{m} - \dot{N}$  is equal to  $\frac{1}{4\pi} (\int_{\Pi} \nu(\psi) dA + \int_S \nu(\psi) dA)$ . Hence, in order to prove monotonicity of  $X$ , we only need to show that  $4\pi \dot{M} \geq -\int_{\Sigma} \nu(\psi) dA$ .

**Conjecture B.2.1.** *Let  $\widetilde{M}$  be an asymptotically flat manifold with positive curvature. Let  $\widetilde{M}$  contain negative point mass singularities  $p_i$  and black holes  $\widetilde{\Sigma}_j$ . Assume that the  $p_i$  can be resolved with a function  $\varphi$  on  $M$  so that  $\varphi = 1$  at infinity,  $\nu(\varphi) = 0$  on  $\Sigma$ ,  $\varphi = 0$  on  $\Pi$ ,  $\nu(\varphi) = -\frac{1}{32N}$  on  $\Pi$ , where  $N$  is the mass of the  $p_i$ , summed appropriately, and  $\Pi$  is minimal. Then*

$$m_{ADM}(\widetilde{M}) \geq \left( \sum_i m_H(\widetilde{\Sigma}_i)^2 \right)^{1/2} - \left( \sum_i |m_R(p_i)|^{2/3} \right)^{3/2}. \quad (\text{B.31})$$

This requires only the following conjecture:

**Conjecture B.2.2.** *Under the conditions of conjecture B.2.1, let  $\psi$  be the harmonic function on  $M$  so that  $\psi = 0$  at infinity,  $\nu(\psi) = -1$  on  $\Sigma$ , and  $\psi = 0$  on  $\Pi$ . Then*

$$\sqrt{4\pi} \left( \int_{\Sigma} \varphi^4 dA \right)^{-1/2} \left( \int_{\Sigma} \psi \varphi^3 dA \right) \geq \int_{\Sigma} \nu(\psi) dA \quad (\text{B.32})$$

Removing this from context gives the following conjecture:

**Conjecture B.2.3.** *Let  $M$  be an asymptotically flat manifold with positive scalar curvature. Let  $M$  have two sets of boundary  $\Sigma$  and  $\Pi$ . Let  $\varphi$  be a function so that  $\varphi = 0$  on  $\Pi$ ,  $\nu(\varphi) = 0$  on  $\Sigma$ ,  $\varphi = 1$  at  $\infty$  and  $\Delta\varphi = 0$ . Furthermore assume that  $\nu(\varphi) = \frac{\sqrt{\pi}}{512|\Pi|^{1/2}}$  on  $\Pi$ . Then let  $\psi$  be the harmonic function defined by  $\psi = 0$  on  $\Pi$ ,  $\psi = 0$  at  $\infty$ . We can choose our boundary conditions for  $\psi$  on  $\Sigma$  so that*

$$0 \geq \sqrt{4\pi} \left( \int_{\Sigma} \varphi^4 dA \right)^{-1/2} \left( \int_{\Sigma} \psi \varphi^3 dA \right) \geq - \int_{\Sigma} \nu(\psi) dA \quad (\text{B.33})$$

*Of course this would require that  $\nu(\psi) > 0$  on  $\Sigma$  and  $\psi < 0$  on  $\Sigma$ .*

If we recast this on the original manifold  $\widetilde{M}$ , we get the following statement:

**Conjecture B.2.4.** *Let  $(\overline{M}, \overline{g})$  be an asymptotically flat 3-manifold, with minimal boundary  $\overline{\Sigma}$  and negative mass singularity  $\overline{p}$  of opposite mass. Also assume that  $(\overline{M}, \overline{g})$  has non-negative scalar curvature. Assume that  $\overline{p}$  can be resolved by a harmonic conformal factor  $\varphi$  so that  $(M, g)$  is asymptotically flat,  $\Sigma$  is still minimal and  $\Pi$ , the resolution of  $p$ , is minimal as well. We can assume that  $\varphi = 1$  at infinity,  $\varphi = 0$  only on  $\Pi$ ,  $\nu(\varphi) = 1$  on  $\Pi$ , and  $\nu(\varphi) = 0$  on  $\Sigma$ .*

*Then there are boundary conditions,  $X$ , on  $\overline{\Sigma}$  so that if  $\psi$  is the solution to  $\overline{\Delta}(\psi/\varphi) = 1$ , with boundary values 0 at  $\infty$  and  $\overline{p}$ , and  $X$  on  $\overline{\Sigma}$ , then*

$$0 \geq \sqrt{\frac{4\pi}{|\overline{\Sigma}|}} \left( \int_{\overline{\Sigma}} \frac{\psi}{\varphi} dA \right) \geq - \int_{\overline{\Sigma}} \frac{\nu(\psi)}{\varphi^4} dA. \quad (\text{B.34})$$

What these  $X$  boundary conditions might be we have been unable to ascertain.

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